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SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S
ELEMENTARY TRIGONOMETRY.



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SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S

ELEMENTARY TRIGONOMETRY

BY

H. S. HALL, M.A.,

FORMERLY SCHOLAR OF CHRIST'S COLLEGE, CAMBRIDGE;
LATE MASTER OF THE MILITARY SIDE, CLIFTON COLLEGE.

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PREFACE.

IN preparing this Key two objects have been kept in view. It is intended first to save the time and lighten the work of teachers, and secondly to afford help to those who study Mathematics without the guidance of a teacher. Accordingly the solutions have generally been given in the most simple and natural manner, with frequent reference to the text and examples in the *Elementary Trigonometry*. In particular, the solutions which involve logarithmic work have been presented in the fullest detail, so that with the help of the Key, a teacher will be able very readily to discover and correct mistakes in the work of his pupils.

For very many of the solutions I am indebted to Mr H. C. Playne of Clifton College, and my thanks are due to him for valuable help all through the book.

H. S. HALL.

January, 1895.

THE present Edition contains solutions of all the examples introduced into the Fourth Edition of the *Elementary Trigonometry*. For many of these I am indebted to Mr H. C. Beaven of Clifton College, whose valuable help I gratefully acknowledge.

H. S. HALL.

October, 1905.

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ELEMENTARY TRIGONOMETRY.

EXAMPLES. I. PAGE 4.

7. $69^\circ 13' 30'' = \cdot7691$ of a right angle = $76^\circ 91' 66\cdot7''$.
8. $19^\circ 0' 45'' = \cdot21125$ of a right angle = $21^\circ 12' 50''$.
9. $50^\circ 37' 5\cdot7'' = \cdot562425$ of a right angle = $56^\circ 24' 25''$.
10. $43^\circ 52' 38\cdot1'' = \cdot487525$ of a right angle = $48^\circ 75' 25''$.
11. $11^\circ 0' 38\cdot4'' = \cdot1223407$ of a right angle = $12^\circ 23' 40\cdot7''$.
12. $142^\circ 15' 45'' = 1\cdot5806944$ of a right angle = $158^\circ 6' 94\cdot4''$.
13. $12' 9'' = \cdot00225$ of a right angle = $22' 50''$.
14. $3' 26\cdot3'' = \cdot000636$ of a right angle = $6^\circ 36\cdot7''$.
15. $56^\circ 87' 50'' = \cdot56875$ of a right angle = $51^\circ 11' 15''$.
16. $39^\circ 6' 25'' = \cdot390625$ of a right angle = $35^\circ 9' 22\cdot5''$.
17. $40^\circ 1' 25\cdot4'' = \cdot4001254$ of a right angle = $36^\circ 0' 40\cdot6''$.
18. $1^\circ 2' 3'' = \cdot010203$ of a right angle = $55' 5\cdot8''$.
19. $3^\circ 2' 55'' = \cdot030205$ of a right angle = $2^\circ 43' 6\cdot4''$.
20. $8^\circ 10' 6\cdot5'' = \cdot0810065$ of a right angle = $7^\circ 17' 26\cdot1''$.
21. $6' 25'' = \cdot000625$ of a right angle = $3' 22\cdot5''$.
22. $37' 5'' = \cdot003705$ of a right angle = $20' 0\cdot4''$.
23. Let the angles expressed in degrees be A and B ;
then $A + B = \frac{9}{10} \times 80^\circ = 72^\circ$, and $A - B = 18^\circ$.
Hence $A = 45^\circ$, $B = 27^\circ$.
24. If n is the number of degrees in the angle, $n + \frac{10}{9}n = 152$; whence $n = 72$.

25. Here $\frac{x}{60}$ = number of degrees, and $\frac{y}{100}$ = number of grades in the angle.
 Therefore $\frac{x}{60} = \frac{9}{10} \cdot \frac{y}{100}$; whence we obtain $50x = 27y$.

26. Here $\frac{s}{60 \times 60}$ = number of degrees, and $\frac{t}{100 \times 100}$ = number of grades in the angle. Therefore $\frac{s}{36} = \frac{9}{10} \times \frac{t}{100}$; that is, $250s = 81t$.

EXAMPLES. II. PAGE 11.

[The following five solutions will sufficiently illustrate this exercise.]

3. From fig. of Art. 17 we have $a^2 = b^2 + c^2 = 400 + 225 = 625$; whence $a = 25$, and $\sin C = \frac{4}{5}$, $\cos B = \frac{4}{5}$, $\cot C = \frac{3}{4}$, $\sec C = \frac{5}{3}$.

6. Let $a = 15$, $b = 9$; then $c^2 = a^2 - b^2 = (a+b)(a-b) = 24 \times 6$; whence $c = 12$, and $\sin C = \frac{4}{5}$, $\cos C = \frac{3}{5}$, $\tan C = \frac{4}{3}$.

8. In the third fig. of Art. 17, let $AC = 41$, $AB = 9$;
 then $BC^2 = 41^2 - 9^2 = (41+9)(41-9)$;
 whence $BC = 40$, and $\sin A = \frac{40}{41}$, $\cot A = \frac{9}{40}$.

10. Here $CD = 2DE$. Hence if $ED = a$, $DC = 2a$, $EC = a\sqrt{5}$. The required ratios may now be written down.

11. From the right-angled $\triangle ABC$ we have $BC = 39$; also from the right-angled $\triangle ACD$ we have $DC = 77$. The required ratios may now be written down.

EXAMPLES. III a. PAGE 17.

Examples 1—25 are too easy to require full solution; the following eight solutions will suffice.

$$10. (1 - \cos^2 \theta) \sec^2 \theta = \sin^2 \theta \times \frac{1}{\cos^2 \theta} = \tan^2 \theta.$$

$$12. \operatorname{cosec} \alpha \sqrt{1 - \sin^2 \alpha} = \frac{1}{\sin \alpha} \times \cos \alpha = \cot \alpha.$$

$$15. (1 - \cos^2 \theta)(1 + \tan^2 \theta) = \sin^2 \theta \sec^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta.$$

$$19. (1 - \cos^2 A)(1 + \cot^2 A) = \sin^2 A \operatorname{cosec}^2 A = 1.$$

20. $\sin \alpha \sec \alpha \sqrt{\operatorname{cosec}^2 \alpha - 1} = \frac{\sin \alpha}{\cos \alpha} \times \cot \alpha = 1.$
22. $\sin^2 \theta \cot^2 \theta + \sin^2 \theta = \sin^2 \theta (1 + \cot^2 \theta) = \sin^2 \theta \operatorname{cosec}^2 \theta = 1.$
23. $(1 + \tan^2 \theta)(1 - \sin^2 \theta) = \sec^2 \theta \cos^2 \theta = 1.$
25. $\operatorname{cosec}^2 \theta \tan^2 \theta - 1 = \frac{1}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} - 1 = \sec^2 \theta - 1 = \tan^2 \theta.$
26. First side = $\cos^2 A + \sin^2 A = 1.$
27. First side = $\sec^2 A - \tan^2 A = 1.$
28. First side = $\sin A \cdot \sin A + \cos A \cdot \cos A = \sin^2 A + \cos^2 A = 1.$
29. First side = $\sec A \cdot \sec A - \tan A \cdot \tan A = \sec^2 A - \tan^2 A = 1.$
30. $\sin^4 \alpha - \cos^4 \alpha = (\sin^2 \alpha + \cos^2 \alpha)(\sin^2 \alpha - \cos^2 \alpha)$
 $= \sin^2 \alpha - \cos^2 \alpha = \sin^2 \alpha - (1 - \sin^2 \alpha)$
 $= 2 \sin^2 \alpha - 1.$
- Also $\sin^2 \alpha - \cos^2 \alpha = 1 - \cos^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha.$
31. First side = $(\sec^2 \alpha - 1)(\sec^2 \alpha + 1) = \tan^2 \alpha (2 + \tan^2 \alpha).$
32. First side = $(\operatorname{cosec}^2 \alpha - 1)(\operatorname{cosec}^2 \alpha + 1) = \cot^2 \alpha (\cot^2 \alpha + 2).$
33. First side = $\left(\frac{\sin \alpha}{\cos \alpha} \cdot \frac{1}{\sin \alpha} \right)^2 - \left(\frac{\sin \alpha}{\cos \alpha} \right)^2 = \sec^2 \alpha - \tan^2 \alpha.$
34. First side = $\left(\frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \right)^2 - \left(\frac{\cos \theta}{\sin \theta} \right)^2 = \operatorname{cosec}^2 \theta - \cot^2 \theta.$
35. $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1 = \sec^2 \theta - \tan^2 \theta.$ Transpose.

EXAMPLES. III b. PAGE 19.

4. $\operatorname{vers} \theta \sec \theta = (1 - \cos \theta) \sec \theta = \sec \theta - 1.$
5. First side = $\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \sin \theta = \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} = \cos \theta.$
6. First side = $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}$
 $= \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta.$
7. First side = $\operatorname{cosec} A \tan A \cos A = \frac{\cos A}{\sin A} \tan A = 1.$
8. First side = $\cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta$
 $= 1 + 1 = 2.$
9. First side = $1 + 2 \tan \theta + \tan^2 \theta + 1 - 2 \tan \theta + \tan^2 \theta$
 $= 2 + 2 \tan^2 \theta = 2(1 + \tan^2 \theta) = 2 \sec^2 \theta.$

10. First side = $\cot^2 \theta - 2 \cot \theta + 1 + \cot^2 \theta + 2 \cot \theta + 1$
 $= 2 \cot^2 \theta + 2 = 2 (\cot^2 \theta + 1) = 2 \operatorname{cosec}^2 \theta.$
11. First side = $\sin^2 A \operatorname{cosec}^2 A + \cos^2 A \sec^2 A = 1 + 1 = 2.$
12. First side = $\cos^2 A \times 1 + \sin^2 A \times 1 = 1.$
13. First side = $\cot^2 \alpha (1 + \cot^2 \alpha) = (\operatorname{cosec}^2 \alpha - 1) \operatorname{cosec}^2 \alpha$
 $= \operatorname{cosec}^4 \alpha - \operatorname{cosec}^2 \alpha.$
14. First side = $\frac{\tan^2 \alpha}{\sec^2 \alpha} \cdot \frac{\operatorname{cosec}^2 \alpha}{\cot^2 \alpha} = \frac{\sin^2 \alpha}{\cos^2 \alpha \sec^2 \alpha} \times \frac{\operatorname{cosec}^2 \alpha \sin^2 \alpha}{\cos^2 \alpha}$
 $= \frac{\sin^2 \alpha}{\cos^2 \alpha} = \sin^2 \alpha \sec^2 \alpha.$
15. First side = $\frac{1 + \sin \alpha + 1 - \sin \alpha}{1 - \sin^2 \alpha} = \frac{2}{\cos^2 \alpha} = 2 \sec^2 \alpha.$
16. First side = $\frac{\tan \alpha (\sec \alpha + 1) + \tan \alpha (\sec \alpha - 1)}{\sec^2 \alpha - 1} = \frac{2 \tan \alpha \sec \alpha}{\tan^2 \alpha}$
 $= \frac{2 \sec \alpha}{\tan \alpha} = \frac{2 \sec \alpha \cos \alpha}{\sin \alpha} = \frac{2}{\sin \alpha} = 2 \operatorname{cosec} \alpha.$
17. First side = $\frac{1}{1 + \sin^2 \alpha} + \frac{1}{\frac{1}{1 + \sin^2 \alpha}} = \frac{1}{1 + \sin^2 \alpha} + \frac{\sin^2 \alpha}{1 + \sin^2 \alpha}$
 $= \frac{1 + \sin^2 \alpha}{1 + \sin^2 \alpha} = 1.$
18. First side = $\left(\frac{1}{\cos \theta} + \frac{1}{\sin \theta} \right) (\sin \theta + \cos \theta)$
 $= \frac{(\sin \theta + \cos \theta)(\sin \theta + \cos \theta)}{\cos \theta \sin \theta}$
 $= \frac{\sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta} = \frac{1 + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta}$
 $= \frac{1}{\cos \theta \sin \theta} + 2 = \sec \theta \cos \theta + 2.$
20. First side = $(1 + \cot \theta)^2 - \operatorname{cosec}^2 \theta = 1 + 2 \cot \theta + \cot^2 \theta - \operatorname{cosec}^2 \theta$
 $= 1 + 2 \cot \theta - 1 = 2 \cot \theta.$
22. First side
 $= \sin^2 A + 2 \sin A \operatorname{cosec} A + \operatorname{cosec}^2 A + \cos^2 A + 2 \cos A \sec A + \sec^2 A$
 $= (\sin^2 A + \cos^2 A) + 2 + (\cot^2 A + 1) + 2 + (\tan^2 A + 1)$
 $= \tan^2 A + \cot^2 A + 7.$
23. First side = $(2 \sec^2 A - 1)(2 \operatorname{cosec}^2 A - 1)$
 $= 4 \sec^2 A \operatorname{cosec}^2 A - 2 \sec^2 A - 2 \operatorname{cosec}^2 A + 1$
 $= 1 + 4 \sec^2 A \operatorname{cosec}^2 A - 2 (\sec^2 A \operatorname{cosec}^2 A) \quad [\text{Art. 31, Ex. 1.}]$
 $= 1 + 2 \sec^2 A \operatorname{cosec}^2 A.$

24. First side = $1 + (\sin^2 A + \cos^2 A) - 2 \sin A + 2 \cos A - 2 \sin A \cos A$
 $= 2[1 - \sin A + \cos A - \sin A \cos A]$
 $= 2(1 - \sin A)(1 + \cos A).$

25. First side = $\sin A \left(1 + \frac{\sin A}{\cos A}\right) + \cos A \left(1 + \frac{\cos A}{\sin A}\right)$
 $= \sin A \frac{(\cos A + \sin A)}{\cos A} + \cos A \frac{(\sin A + \cos A)}{\sin A}$
 $= \tan A (\sin A + \cos A) + \cot A (\sin A + \cos A)$
 $= (\sin A + \cos A) \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$
 $= \frac{(\sin A + \cos A)(\sin^2 A + \cos^2 A)}{\sin A \cos A}$
 $= \frac{\sin A + \cos A}{\sin A \cos A} = \frac{1}{\cos A} + \frac{1}{\sin A} = \sec A + \operatorname{cosec} A.$

26. First side = $\cos \theta (2 \tan^2 \theta + 5 \tan \theta + 2) = \frac{2 \sin^2 \theta}{\cos^2 \theta} \cos \theta + 5 \sin \theta + 2 \cos \theta$
 $= 2 \left(\frac{\sin^2 \theta}{\cos \theta} + \cos \theta \right) + 5 \sin \theta = \frac{2(\sin^2 \theta + \cos^2 \theta)}{\cos \theta} + 5 \sin \theta$
 $= 2 \sec \theta + 5 \sin \theta.$

27. First side = $\left(\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \right)^2 = \left(\frac{1 + \sin \theta}{\cos \theta} \right)^2$
 $= \frac{(1 + \sin \theta)(1 + \sin \theta)}{1 - \sin^2 \theta} = \frac{(1 + \sin \theta)(1 + \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)} = \frac{1 + \sin \theta}{1 - \sin \theta}.$

28. First side = $\frac{\cos \theta (2 \sin \theta - 1)}{2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (2 \sin \theta - 1)}{\sin \theta (2 \sin \theta - 1)} = \cot \theta.$

29. First side = $\frac{\cot^2 \theta (\sec \theta - 1)}{1 + \sin \theta} - \frac{\sec^2 \theta (1 - \sin \theta)}{1 + \sec \theta}$
 $= \frac{\cot^2 \theta (\sec^2 \theta - 1) - \sec^2 \theta (1 - \sin^2 \theta)}{(1 + \sin \theta)(1 + \sec \theta)}$
 $= \frac{\cot^2 \theta \tan^2 \theta - \sec^2 \theta \cos^2 \theta}{(1 + \sin \theta)(1 + \sec \theta)} = 0.$

30. $\tan^2 \alpha + \sec^2 \beta = (\sec^2 \alpha - 1) + (\tan^2 \beta + 1) = \sec^2 \alpha + \tan^2 \beta.$

31. $\frac{\tan \alpha + \cot \beta}{\cot \alpha + \tan \beta} = \frac{\tan \alpha + \frac{1}{\tan \beta}}{\frac{1}{\tan \alpha} + \tan \beta} = \frac{\tan \alpha \tan \beta + 1}{\tan \beta} \cdot \frac{\tan \alpha}{\tan \alpha \tan \beta + 1} = \frac{\tan \alpha}{\tan \beta}.$

33. First side = $\cot \alpha \tan \alpha \tan \beta + \tan \beta \cot \beta \cot \alpha$
 $= \tan \beta + \cot \alpha.$

34. First side = $\sin^2 \alpha (1 - \sin^2 \beta) - (1 - \sin^2 \alpha) \sin^2 \beta$
 $= \sin^2 \alpha - \sin^2 \alpha \sin^2 \beta - \sin^2 \beta + \sin^2 \alpha \sin^2 \beta$
 $= \sin^2 \alpha - \sin^2 \beta$.
35. First side = $(1 + \tan^2 \alpha) \tan^2 \beta - \tan^2 \alpha (1 + \tan^2 \beta)$
 $= \tan^2 \beta + \tan^2 \alpha \tan^2 \beta - \tan^2 \alpha - \tan^2 \alpha \tan^2 \beta$
 $= \tan^2 \beta - \tan^2 \alpha$.
36. First side = $\sin^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta$,
the other terms cancelling;
this expression = $(\sin^2 \alpha + \cos^2 \alpha) \cos^2 \beta + (\cos^2 \alpha + \sin^2 \alpha) \sin^2 \beta$
 $= \cos^2 \beta + \sin^2 \beta = 1$.

EXAMPLES. III. c. PAGE 23.

1. $\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}} = \frac{1}{\sqrt{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}}$.

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$$
.

2. $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$ [Art. 32, Ex. 1]

$$= \frac{4}{3} \div \sqrt{1 + \frac{16}{9}} = \frac{4}{3} \times \frac{3}{5} = \frac{4}{5}$$
.

$$\cos A = \cot A \cdot \sin A = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}$$
.

5. $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{49}} = \sqrt{\frac{48}{49}} = \frac{\sqrt{48}}{7}$.

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{7} \div \frac{\sqrt{48}}{7} = \frac{1}{\sqrt{48}}$$
.

6. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}$. Therefore $\sec A = \frac{25}{24}$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}$$
.

8. $\operatorname{cosec} \alpha = \sqrt{1 + \cot^2 \alpha}$. [Art. 27.]

$$\cos \alpha = \cot \alpha \sin \alpha = \frac{\cot \alpha}{\operatorname{cosec} \alpha} = \frac{\cot \alpha}{\sqrt{1 + \cot^2 \alpha}}$$
.

$$10. \quad \text{cosec } A = \frac{1}{\sin A}; \quad \cos A = \sqrt{1 - \sin^2 A}; \quad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}; \quad \cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

11. Here $\sin A = \cos A$, so that $\tan A = 1$.

$$\therefore \text{cosec } A = \sqrt{1 + \cot^2 A} = \sqrt{1 + 1} = \sqrt{2}.$$

$$12. \quad \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{m}{n} \div \sqrt{1 - \frac{m^2}{n^2}}$$

$$= \frac{m}{n} \times \frac{n}{\sqrt{n^2 - m^2}} = \frac{m}{\sqrt{n^2 - m^2}}.$$

$$13. \quad p^2 \cot^2 \theta = q^2 - p^2; \quad \therefore p^2 (\cot^2 \theta + 1) = q^2.$$

$$\therefore \quad p^2 \text{cosec}^2 \theta = q^2, \text{ so that } \sin \theta = \frac{p}{q}.$$

14. In the diagram of Ex. 2, Art. 33, let $PQ = 2m$, $PR = m^2 + 1$; then

$$RQ^2 = (m^2 + 1)^2 - (2m)^2 = (m^2 - 1)^2.$$

$$\therefore \quad RQ = m^2 - 1.$$

$$\therefore \quad \tan A = \frac{m^2 - 1}{2m}, \quad \sin A = \frac{m^2 - 1}{m^2 + 1}.$$

$$16. \quad \text{The expression} = \frac{2 \tan \alpha - 3}{4 \tan \alpha - 9}; \text{ but } \tan \alpha = \sqrt{\frac{169}{25} - 1} = \frac{12}{5}.$$

$$\therefore \quad \text{the expression} = \frac{\frac{2 \cdot 12}{5} - 3}{\frac{4 \cdot 12}{5} - 9} = \frac{9}{3} = 3.$$

$$17. \quad \text{The expression} = \frac{p \cot \theta - q}{p \cot \theta + q} = \frac{\frac{p^2 - q^2}{p^2 + q}}{q} = \frac{p^2 - q^2}{p^2 + q^2}.$$

EXAMPLES. IV. a. PAGE 26.

Let E stand for the expression to be evaluated in each case; then

$$6. \quad E = (1)^2 \times \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} (\sqrt{3})^2 = \frac{3}{2}.$$

$$7. \quad E = (\sqrt{3})^2 + 4 \left(\frac{1}{\sqrt{2}} \right)^2 + 3 \left(\frac{2}{\sqrt{3}} \right)^2 = 3 + \frac{4}{2} + 4 = 9.$$

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8. $E = \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right)^2 + (\sqrt{2})^2 - 2 \left(\frac{1}{\sqrt{3}} \right)^2 = \frac{1}{2} \cdot \frac{4}{3} + 2 - \frac{2}{3} = 2.$

9. $E = \left(\frac{1}{\sqrt{3}} \right)^2 + 2 \left(\frac{\sqrt{3}}{2} \right)^2 + 1 - \sqrt{3} + \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{3} + \sqrt{3} + 1 - \sqrt{3} + \frac{3}{4} = \frac{25}{12}.$

10. $E = (1)^2 + \frac{1}{2} - \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{3}{4} \left(\frac{1}{\sqrt{3}} \right)^2 = 1 + \frac{1}{2} - \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$

11. $E = 3 \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (\sqrt{2})^2 - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 = 1 + 1 - 1 - \frac{1}{4} = \frac{3}{4}.$

12. $E = \frac{1}{2} - 1^2 + \frac{3}{4} \left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} = \frac{1}{2} - 1 + \frac{1}{4} + \frac{3}{4} - \frac{1}{2} = 0.$

13. $E = \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} \cdot 2 \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left(\frac{1}{\sqrt{2}} \right)^2 (\sqrt{3})^2 = \frac{1}{4} - \frac{1}{3} + 2 = 1\frac{1}{2}.$

14. We have

$$(1)^2 - \left(\frac{1}{2} \right)^2 = x \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{3}; \quad x \cdot \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{(\sqrt{3})^2 \cdot 2 \cdot 1}{(\sqrt{2})^2 2};$$

$$1 - \frac{1}{4} = x \cdot \frac{\sqrt{3}}{2}; \quad \frac{x}{4} = \frac{3}{2};$$

$$\therefore x = \frac{\sqrt{3}}{2}. \quad \therefore x = 6.$$

EXAMPLES. IV. b. PAGE 28.

For Examples 9—14, see Example 1, page 28.

20. Second side $= 1 + \sin^2 A \sec^2 A = 1 + \tan^2 A = \sec^2 A = \operatorname{cosec}^2 (90^\circ - A).$

21. First side $= \sin A \cot A \tan A \operatorname{cosec} A = 1.$

22. First side $= \operatorname{cosec} A - \cot A \sin A \cot A = \operatorname{cosec} A (1 - \cos^2 A) = \sin A.$

23. First side $= \tan^2 A \operatorname{cosec}^2 A - \sin^2 A \sec^2 A$
 $= \sec^2 A - \tan^2 A = 1.$

24. First side

$$= \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \operatorname{cosec} A \sec A = \operatorname{cosec} A \operatorname{cosec} (90^\circ - A).$$

25. First side $= \frac{\cos A}{\operatorname{cosec} A} \cdot \frac{\cot A}{\cos A} = \cos A.$

26. First side $= \frac{\operatorname{cosec}^2 A \tan^2 A}{\tan A} \cdot \frac{\cot A}{\sec^2 A} = \cot^2 A.$
 $= \operatorname{cosec}^2 A - 1 = \sec^2 (90^\circ - A) - 1.$

27. First side = $\frac{\tan A}{\operatorname{cosec}^2 A} \cdot \frac{\sec A \cot^3 A}{\cos^2 A} = \sec A = \sqrt{\tan^2 A + 1}$.

28. First side = $\frac{\sin^2 A}{1 - \cos A} = \frac{1 - \cos^2 A}{1 - \cos A} = 1 + \cos A = 1 + \sin (90^\circ - A)$.

29. First side = $\frac{\cot^2 A \cos^2 A}{\cot A (1 + \sin A)} = \cot A (1 - \sin A) = \tan (90^\circ - A) - \cos A$.

30. $x \cos A \tan A = \sin A$; 31. $\sec^2 A - x \tan A = 1$;

$$\therefore x = 1.$$

$$\therefore \tan^2 A = x \tan A ;$$

$$\therefore x = \tan A.$$

EXAMPLES. IV. d. PAGE 31.

9. $1 + \tan^2 \theta = 2 \tan^2 \theta$;
 $\tan^2 \theta = 1$;

$$\therefore \tan \theta = \pm 1$$
 ;

$$\therefore \theta = 45^\circ$$
.

10. $1 + \cot^2 \theta = 4 \cot^2 \theta$;
 $3 \cot^2 \theta = 1$;

$$\therefore \cot \theta = \pm \frac{1}{\sqrt{3}}$$
 ;

$$\therefore \theta = 60^\circ$$
.

11. $1 + \tan^2 \theta = 3 \tan^2 \theta - 1$;
 $2 \tan^2 \theta = 2$;
 $\therefore \tan \theta = \pm 1$;
 $\therefore \theta = 45^\circ$.

12. $1 + \tan^2 \theta + \tan^2 \theta = 7$;
 $2 \tan^2 \theta = 6$;
 $\therefore \tan \theta = \pm \sqrt{3}$;
 $\therefore \theta = 60^\circ$.

13. $\cot^2 \theta + 1 + \cot^2 \theta = 3$;
 $\cot^2 \theta = 1$;
 $\therefore \cot \theta = \pm 1$;
 $\therefore \theta = 45^\circ$.

14. $2(\cos^2 \theta - 1 + \cos^2 \theta) = 1$;
 $4 \cos^2 \theta = 3$;
 $\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$;
 $\therefore \theta = 30^\circ$.

15. $2 \cos^2 \theta + 4 - 4 \cos^2 \theta = 3$;
 $2 \cos^2 \theta = 1$;
 $\therefore \cos \theta = \pm \frac{1}{\sqrt{2}}$;
 $\therefore \theta = 45^\circ$.

16. $6 \cos^2 \theta - \cos \theta - 1 = 0$;
 $(3 \cos \theta + 1)(2 \cos \theta - 1) = 0$;
 $\therefore \cos \theta = \frac{1}{2}$ or $-\frac{1}{3}$;
 $\therefore \theta = 60^\circ$.

17. $12 \sin^2 \theta - 4 \sin \theta - 1 = 0$;
 $(6 \sin \theta + 1)(2 \sin \theta - 1) = 0$;
 $\therefore \sin \theta = \frac{1}{2}$ or $-\frac{1}{6}$;
 $\therefore \theta = 30^\circ$.

18. $2 - 2 \cos^2 \theta = 3 \cos \theta$;
 $2 \cos^2 \theta + 3 \cos \theta - 2 = 0$;
 $\therefore (2 \cos \theta - 1)(\cos \theta + 2) = 0$;
 $\therefore \cos \theta = \frac{1}{2}$, so that $\theta = 60^\circ$.

19. $\tan \theta = 4 - \frac{3}{\tan \theta};$

$$\tan^2 \theta - 4 \tan \theta + 3 = 0;$$

$$\therefore (\tan \theta - 1)(\tan \theta - 3) = 0;$$

$$\therefore \tan \theta = 1 \text{ or } 3;$$

$$\therefore \theta = 45^\circ \text{, or } 71^\circ 34'.$$

21. $\frac{1 + \tan^2 \theta}{\tan \theta} = 2 \sec \theta;$

$$\frac{\sec^2 \theta}{\tan \theta} = 2 \sec \theta;$$

$$\therefore \sec \theta = 0, \text{ or } \frac{\sec \theta}{\tan \theta} = 2;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

23. $\frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta};$

$$\sin^2 \theta - \cos^2 \theta = \cos \theta;$$

$$1 - 2 \cos^2 \theta = \cos \theta;$$

$$2 \cos^2 \theta + \cos \theta - 1 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 1) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ or } -1;$$

$$\therefore \theta = 60^\circ.$$

25. $\tan \theta (2 \sin \theta - 1) = 2 \sin \theta - 1;$
 $(2 \sin \theta - 1)(\tan \theta - 1) = 0;$

$$\therefore \sin \theta = \frac{1}{2}, \text{ or } \tan \theta = 1;$$

$$\therefore \theta = 30^\circ \text{ or } 45^\circ.$$

27. $5 \tan \theta + \frac{6}{\tan \theta} = 11;$

$$5 \tan^2 \theta - 11 \tan \theta + 6 = 0;$$

$$(5 \tan \theta - 6)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } 1 \cdot 2;$$

$$\therefore \theta = 45^\circ, \text{ or } 50^\circ 12'.$$

20. $\cos^2 \theta - 1 + \cos^2 \theta = 2 - 5 \cos \theta;$

$$2 \cos^2 \theta + 5 \cos \theta - 3 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ so that } \theta = 60^\circ.$$

22. $\frac{4}{\sin \theta} + 2 \sin \theta = 0;$

$$2 \sin^2 \theta - 9 \sin \theta + 4 = 0;$$

$$\therefore (2 \sin \theta - 1)(\sin \theta - 4) = 0;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

24. $2 \cos \theta + 2 \sqrt{2} = \frac{3}{\cos \theta};$

$$2 \cos^2 \theta + 2 \sqrt{2} \cos \theta - 3 = 0;$$

$$\therefore (\sqrt{2} \cos \theta - 1)(\sqrt{2} \cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{\sqrt{2}};$$

$$\therefore \theta = 45^\circ.$$

26. $6 \frac{\sin \theta}{\cos \theta} - \frac{5\sqrt{3}}{\cos \theta} + 12 \frac{\cos \theta}{\sin \theta} = 0;$

$$6 \sin^2 \theta - 5\sqrt{3} \sin \theta + 12(1 - \sin^2 \theta) = 0;$$

$$6 \sin^2 \theta + 5\sqrt{3} \sin \theta - 12 = 0;$$

$$\therefore (2 \sin \theta - \sqrt{3})(3 \sin \theta + 4\sqrt{3}) = 0;$$

$$\therefore \sin \theta = \frac{\sqrt{3}}{2}, \text{ so that } \theta = 60^\circ.$$

28. $1 + \tan^2 \theta + \tan^2 \theta = 3 \tan \theta;$

$$2 \tan^2 \theta - 3 \tan \theta + 1 = 0;$$

$$\therefore (2 \tan \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } \frac{1}{2}.$$

$$\therefore \theta = 45^\circ, \text{ or } 26^\circ 34'.$$

MISCELLANEOUS EXAMPLES. A. PAGE 32.

3. If θ be the angle, we have $\sin \theta = \frac{21}{29}$, so that $\operatorname{cosec} \theta = \frac{29}{21}$.

$$\text{Also } \cos \theta = \sqrt{1 - \left(\frac{21}{29}\right)^2} = \frac{\sqrt{(29+21)(29-21)}}{29} = \frac{20}{29}.$$

$$4. \tan A = \frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}} = \frac{15}{\sqrt{2 \times 32}} = \frac{15}{8};$$

$$\sec A = \sqrt{1 + \tan^2 A} = \frac{\sqrt{8^2 + 15^2}}{8} = \frac{17}{8}.$$

5. First side = $\operatorname{cosec}^2 A - \cot^2 A - 1 = 0$.

7. $b = \sqrt{a^2 + c^2} = \sqrt{1681} = 41$.

$$\cot A = \frac{c}{a} = \frac{9}{40}; \quad \sec A = \frac{b}{c} = \frac{41}{9}; \quad \sec C = \frac{b}{a} = \frac{41}{40}.$$

8. See Article 16.

9. First side = $\cos \theta (1 - \cos \theta) \frac{1 + \cos \theta}{\cos \theta} = 1 - \cos^2 \theta = \sin^2 \theta$.

10. We have $\sec^2 a = 1 + \tan^2 a = \frac{1 + \cot^2 a}{\cot^2 a}; \quad \therefore \sec a = \sqrt{\frac{1 + \cot^2 a}{\cot a}}$.

Also $\operatorname{cosec}^2 a = 1 + \cot^2 a$; so that $\operatorname{cosec} a = \sqrt{1 + \cot^2 a}$.

11. First side = $3\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{4} \cdot 2 + 5 \cdot 1 - \frac{2}{3} \cdot \left(\frac{\sqrt{3}}{2}\right)^2 = 1 + \frac{1}{2} + 5 - \frac{1}{2} = 6$.

12. $\sin a = \frac{1}{\sqrt{1 + \cot^2 a}} = \frac{m}{\sqrt{m^2 + n^2}}$; $\sec a = \sqrt{1 + \tan^2 a} = \frac{\sqrt{m^2 + n^2}}{n}$.

13. m sexagesimal minutes = $\frac{m}{60 \times 90}$ right angles,

$$n$$
 centesimal minutes = $\frac{n}{100 \times 100}$ right angles.

$$\therefore \frac{m}{60 \times 90} = \frac{n}{100 \times 100}; \quad \text{whence } m = .54n.$$

14. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{16}{25}} = +\frac{3}{5}$, since A is acute;

$$\therefore \tan A + \sec A = \frac{4}{3} + \frac{5}{3} = 3.$$

15. First side = $\tan A \cot A \sin A \cot A = \cos A$.

16. $RQ = \sqrt{20^2 + 21^2} = \sqrt{841} = 29$:

$$\therefore \tan Q = \frac{RP}{PQ} = \frac{20}{21}; \quad \operatorname{cosec} Q = \frac{QR}{RP} = \frac{29}{20}.$$

17. First side = $\frac{\sin^2 \alpha - \cos^2 \alpha}{\sin \alpha \cos \alpha}$. $\sin \alpha \cos \alpha = 1 - \cos^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha$.

18. See Art. 39, Ex. 1.

19. Second side = $(\sqrt{3})^2 - 2 \cdot \left(\frac{1}{2}\right)^2 - \frac{3}{4} (\sqrt{2})^2 = 3 - \frac{1}{2} - \frac{3}{2} = 3 - 2$
 $= \tan^2 60^\circ - 2 \tan^2 45^\circ$.

20. (1) $3 \sin \theta = 2 \cos^2 \theta$; (2) $5 \tan \theta - \sec^2 \theta = 3$;
 $2 \sin^2 \theta + 3 \sin \theta - 2 = 0$; $5 \tan \theta - 1 - \tan^2 \theta = 3$;
 $\therefore (2 \sin \theta - 1)(\sin \theta + 2) = 0$; $\tan^2 \theta - 5 \tan \theta + 4 = 0$;
 $\therefore \sin \theta = \frac{1}{2}$, so that $\theta = 30^\circ$. $\therefore (\tan \theta - 1)(\tan \theta - 4) = 0$;
whence $\theta = 45^\circ$, or $75^\circ 58'$.

21. First side = $1 - (\sec^2 A - \tan^2 A)^2 = 1 - 1 = 0$.

22. $6 \sin^2 \theta - 11 \sin \theta + 4 = 0$; $\therefore (2 \sin \theta - 1)(3 \sin \theta - 4) = 0$;
 $\therefore 2 \sin \theta - 1 = 0$; whence $\theta = 30^\circ$;
or $3 \sin \theta - 4 = 0$; whence $\sin \theta = \frac{4}{3}$, which is impossible.

23. $\tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$.

24. $c + c^{-1} = \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \sec A \operatorname{cosec} A$.

25. $(\sin \theta + 2)(3 \sin \theta - 1) = 0$, whence $\sin \theta = \frac{1}{3} = .3333$;
 $\therefore \theta = 19^\circ 28'$.

EXAMPLES. V. a. PAGE 37.

1. $c = \sqrt{a^2 - b^2} = \sqrt{16 - 12} = 2$.

$$\sin C = \frac{c}{a} = \frac{1}{2}; \therefore C = 30^\circ. \quad \sin B = \frac{b}{a} = \frac{\sqrt{3}}{2}; \therefore B = 60^\circ.$$

2. $a = \sqrt{b^2 - c^2} = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}$.

$$\sin C = \frac{c}{b} = \frac{1}{2}; \therefore C = 30^\circ; \text{ and } A = 90^\circ - C = 60^\circ.$$

3. $c = \sqrt{a^2 + b^2} = \sqrt{144 + 48} = \sqrt{192} = 8\sqrt{3}$.

$$\sin B = \frac{b}{c} = \frac{12}{8\sqrt{3}} = \frac{\sqrt{3}}{2}; \therefore B = 60^\circ; \text{ and } A = 90^\circ - B = 30^\circ.$$

4. $c = \sqrt{a^2 - b^2} = \sqrt{90 \times 30} = 30\sqrt{3}$.

$$\sin C = \frac{c}{a} = \frac{\sqrt{3}}{2}; \therefore C = 60^\circ, B = 30^\circ.$$

6. $c = \sqrt{a^2 + b^2} = \sqrt{75 + 3 \times 75} = 10\sqrt{3}.$

$$\sin B = \frac{b}{c} = \frac{\sqrt{3}}{2}; \quad \therefore B = 60^\circ, A = 30^\circ.$$

7. $b = c = 2; \quad \therefore B = C = 45^\circ.$

$$a = \sqrt{b^2 + c^2} = 2\sqrt{2}.$$

8. $a = \sqrt{b^2 - c^2} = \sqrt{3 \times 36 - 27} = 9.$

$$\sin C = \frac{c}{b} = \frac{1}{2}; \quad \therefore C = 30^\circ, B = 60^\circ.$$

9. $B = 90^\circ - A = 60^\circ.$

$$\frac{b}{a} = \tan B; \quad \therefore b = 9\sqrt{3} \cdot \sqrt{3} = 27.$$

$$\frac{c}{a} = \sec B; \quad \therefore c = 9\sqrt{3} \cdot 2 = 18\sqrt{3}.$$

10. $C = 90^\circ - 25^\circ = 65^\circ.$

$$b = a \cos C = 4 \times .4226;$$

$$c = a \sin C = 4 \times .9063.$$

11. $B = 90^\circ - A = 36^\circ.$

$$a = c \cos B = 8 \times .8090;$$

$$b = c \sin B = 8 \times .5878.$$

12. $B = 180^\circ - C - A = 90^\circ.$

$$a = b \cos C = 6 \times .4540;$$

$$c = b \sin 63^\circ = 6 \times .8910.$$

13. $A = 90^\circ - C = 53^\circ.$

$$a = b \cos C = 100 \times .7986;$$

$$c = b \sin C = 100 \times .6018.$$

14. $C = 180^\circ - A - B = 90^\circ.$

$$a = b \tan A = 20; \quad c = b \sec A = 40.$$

15. $A = 180^\circ - B - C = 90^\circ.$

$$b = c = 4; \quad a = \sqrt{b^2 + c^2} = 4\sqrt{2}.$$

16. $A = 180^\circ - B - C = 90^\circ.$

$$b = a \cos C = 4; \quad c = a \sin C = 4\sqrt{3}.$$

17. $a = b \tan A = \frac{49}{.07} = 700.$

18. $a = c \sin A = 50 \times .62 = 31.$

19. $c = a \tan C = 100 \times .8647 = 86.47.$

20. $a = b \sec C = 200 \times 4.89 = 978.$

21. $C = 90^\circ - A = 54^\circ.$

$$a = c \tan A = 100 \times .73 = 73;$$

$$b = c \sec A = 100 \times 1.24 = 124.$$

22. $\sin C = \frac{c}{a} = .37; \quad \therefore C = 21^\circ 43'.$

$$B = 90^\circ - C = 68^\circ 17';$$

$$b = a \cos C = 100 \times .93 = 93.$$

24. $c = \sqrt{a^2 + b^2} = \sqrt{124609} = 353.$

23. $C = 90^\circ - B = 50^\circ 36'.$

$$c = b \cot B = 25 \times 1.2174 = 30.435;$$

$$a = b \operatorname{cosec} B = 25 \times 1.5755 = 34.3875.$$

$$\tan B = \frac{b}{a} = \frac{272}{225} = 1.209;$$

$$\therefore B = 50^\circ 24'.$$

$$A = 90^\circ - B = 39^\circ 36'.$$

25. $\cos A = \frac{22.75}{25} = .91; \quad \text{whence } A = 24^\circ 30'. \quad \text{Hence } B = 65^\circ 30'.$

$$a = c \sin A = 25 \times .4147 = 10.37, \text{ approx.}$$

EXAMPLES. V. b. PAGE 39.

- $P = 180^\circ - 30^\circ - 120^\circ = 30^\circ = A$; $c = BD \cosec 30^\circ = 20$;
 $\therefore CB = CA = 20$; $a = BD \cosec 45^\circ = 10\sqrt{2}$.
 $\therefore BD = BC \sin 60^\circ = 10\sqrt{3}$.
 - Since $B + C = 90^\circ$; $\therefore A = 90^\circ$. $AB = BD \sec 30^\circ = 10\sqrt{3}$ ft.
 $AC = AB \tan 30^\circ = 10$ ft. $AD = AB \sin 30^\circ = 5\sqrt{3}$ ft.
 - Let QS be the perpendicular from Q on PR .
Then $PR = 8 \sec 60^\circ = 16$. $SR = 8 \cos 60^\circ = 4$. $\therefore SP = 16 - 4 = 12$
 - $SQ = 36 \tan 53^\circ = 47.77$. $RQ = 36 \tan 35^\circ = 25.21$.
 $\therefore RS = SQ - RQ = 22.56$.
 - We have $\angle PRQ = 180^\circ - 135^\circ = 45^\circ$; $\therefore \angle QPR = 45^\circ$;
 $\therefore QR = QP = 20$, and $\angle QPS = 90^\circ - 25^\circ = 65^\circ$,
 $\therefore SQ = 20 \tan 65^\circ = 42.89$, $\therefore RS = 42.89 - 20 = 22.89$.
 - Let AD be the perpendicular and let $AD = x$.
Then $\angle BAD = 90^\circ - 45^\circ = 45^\circ = \angle ABD$, $\therefore DB = DA = x$.
Now $\frac{DA}{DC} = \tan 60^\circ$; $\therefore \frac{x}{x-40} = \sqrt{3}$; $\therefore x(\sqrt{3}-1) = 40\sqrt{3}$.
 $\therefore x = \frac{40\sqrt{3}}{\sqrt{3}-1} = 20\sqrt{3}(\sqrt{3}+1) = 20(3+\sqrt{3})$.
 $\therefore \text{perpendicular} = 20(3+\sqrt{3}) = 94.64$.
 - Let $DC = x$.
Since $\angle DCB = 90^\circ - 45^\circ = 45^\circ = \angle CBD$; $\therefore DB = DC = x$.
And $\frac{DA}{DC} = \cot 35^\circ 18'$; $\therefore \frac{x+41.24}{x} = 1.4124$.
 $\therefore x+41.24=1.4124x$; $\therefore x=100$;
that is, $DC = DB = 100$.
 - The perp. $AD = 20 \sin 42^\circ = 20 \times 0.6691 = 13.382$,
 $\tan C = \frac{AD}{CD} = \frac{13.382}{18.138} = 0.7378$;
whence $C = 36^\circ 25'$.

EXAMPLES. VI. a. PAGE 42.

For Examples 1—5 see figure on page 40.

1. Let BC = height of chimney, $AC = 300$ ft,
then elevation = $\angle BAC = 30^\circ$, $\therefore BC = AC \tan 30^\circ = 100\sqrt{3} = 173.2$ ft.

2. Let B be the top of the mast, $BC=160$ feet, A the boat observed.

Then $\angle BAC=30^\circ$, $\angle ABC=60^\circ$.

\therefore distance required $= AC = BC \tan 60^\circ = 160\sqrt{3} = 277.12$ ft.

3. Let BC represent the pole, and AC its shadow.

Then $\tan A = \frac{BC}{AC} = \frac{6}{2\sqrt{3}} = \sqrt{3}$; \therefore angle of elevation $= 60^\circ$.

4. Let BC represent the tower; A the position of the observer.

Then $AC=86.6$ ft. and $\angle BAC=30^\circ$.

\therefore height of tower $= AC \tan 30^\circ = \frac{86.6}{\sqrt{3}} = 50$ ft.

Distance $AB = BC \operatorname{cosec} 30^\circ = 2BC = 100$ ft.

5. Let AB represent the ladder, and BC the wall.

Then $AB=45$ ft. $\angle ABC=60^\circ$.

\therefore height of wall $= BC = AB \cos 60^\circ = 22.5$ ft.

Distance $AC = AB \sin 60^\circ = \frac{45\sqrt{3}}{2} = 38.97$ ft.

6. See figure on page 9.

Let DE , BC represent the masts, and OCE the horizon.

Then $\angle BOC=33^\circ 41'$. $BC=40$ ft. $DE=60$ ft.

And $OC=BC \cot 33^\circ 41'=60$ ft. $OE=DE \cot 33^\circ 41'=90$ ft.

\therefore distance required $= OE - OC = 30$ ft.

7. See figure on page 40.

Let BC represent the cliff and A the observer.

Then $\angle BAC=41^\circ 18'$. $BC=132$ yds.

\therefore distance required $= AB = BC \operatorname{cosec} 41^\circ 18' = \frac{132}{.66} = 200$ yds.

8. See figure on page 9.

Let BC , DE represent the chimneys, and O the observer.

Then $OC=100$ yds. $\angle BOC=27^\circ 2'$.

Now $BC=OC \tan 27^\circ 2'=51$ yds.

$\therefore DE=BC+30$ yds. $= 81$ yds.

9. See figure on page 41.

Let PT be the tower, and Q , R the two points of observation.

Then $\angle PQR=30^\circ$, $\angle PRT=60^\circ$, $QR=100$ yds.

$\therefore \angle RPQ=60^\circ - 30^\circ = 30^\circ = \angle PQR$, $\therefore RP=RQ=100$ yds.

\therefore height of tower $= RP \sin 60^\circ = 50\sqrt{3} = 86.6$ yds.

10. Let AB be the flagstaff, BC the building, D the point of observation.

Then $DC = 40$ ft. $\angle ADC = 60^\circ$, $\angle BDC = 30^\circ$;

$$\therefore \angle DAB = \angle ADB = 30^\circ; \quad \therefore BA = BD = 40 \sec 30^\circ = \frac{80}{\sqrt{3}} = 49.19 \text{ ft.}$$

11. See figure on page 41.

Let PT be the spire, R, Q the two points of observation.

Then $QR = 200$ ft. $\angle PRT = 45^\circ$, $\angle PQR = 30^\circ$.

$$\angle RPT = 45^\circ; \quad \therefore TP = TR.$$

Let x ft. = height of spire.

$$\text{Then } \frac{x}{x+200} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\therefore x = \frac{200}{\sqrt{3}-1} = 100(1+\sqrt{3}) = 273.2 \text{ feet.}$$

12. See figure on page 43.

Let CD represent the post, and AB the steeple.

Then $CD = 30$ ft. $\angle ACE = 30^\circ$, $\angle ADB = 45^\circ$;

$\therefore \angle DAB = \angle ADB = 45^\circ$; $\therefore BA = BD = x$ feet, say.

$$\therefore \tan 30^\circ = \frac{AE}{EC} = \frac{x-30}{x}; \quad \therefore x(\sqrt{3}-1) = 30\sqrt{3}. \quad \therefore x = 70.98 \text{ ft.}$$

That is height = distance = 70.98 ft.

13. Let B be the top of the hill, and C the point on the horizontal plane vertically below B . Let D be the position of the balloon when the observation is made. Draw DE perpendicular to BC .

Then $BC = 3300$ ft. $\angle BDE = 30^\circ$, $\angle BAC = 60^\circ$;

$$\therefore AC = BC \cot 60^\circ = 1100\sqrt{3} \text{ feet.}$$

And $BE = DE \tan 30^\circ = AC \tan 30^\circ = 1100$ feet.

$$\therefore DA = EC = 3300 - 1100 = 2200 \text{ feet.}$$

\therefore the balloon rises 2200 feet in 5 minutes,

that is, $\frac{2200 \times 60}{1760 \times 3 \times 5}$ miles per hour, or 5 miles per hour.

14. See figure on page 44.

Let OA represent the monument, B, C the two objects, OP the horizontal line through O ;

Then $\angle POC = 30^\circ$; $\therefore \angle OCB = 30^\circ$.

$$\angle POB = 45^\circ; \quad \therefore \angle BOA = \angle OBA = 45^\circ; \quad \therefore AO = AB = 100 \text{ feet.}$$

Let x feet = CB = required distance.

$$\text{Then } \frac{100}{x+100} = \tan 30^\circ = \frac{1}{\sqrt{3}}; \quad \therefore x = 100(\sqrt{3}-1);$$

\therefore distance required = 73.2 feet.

15. See figure on page 43.

Let AB represent the monument, CD the tower.

Then $AB = 96$ feet, and the angles are as in the figure;

$$\therefore DB = AB \cot 60^\circ = 32\sqrt{3} \text{ feet};$$

$$\therefore AE = CE \tan 30^\circ = DB \tan 30^\circ = 32 \text{ feet}.$$

$$\therefore \text{height of tower} = CD = EB = 96 - 32 = 64 \text{ feet}.$$

16. See figure on page 44.

Let OA represent the cliff, B, C the two boats.

Then $OA = 150$ ft., $\angle OBA = 30^\circ$, $\angle OCB = 15^\circ$;

$$\therefore \angle BOC = 15^\circ, \text{ and } BC = BO.$$

$$\therefore \text{required distance} = BC = BO = AO \operatorname{cosec} 30^\circ = 300 \text{ ft.}$$

17. See figure on page 44.

Let O represent the top of the hill, B, C the milestones.

Then $\angle OBA = 45^\circ$, $\angle OCB = 22^\circ$; $\therefore \angle BOA = 45^\circ$, so that $AO = AB$.

Let $AO = AB = x$ yards.

Then $\frac{AC}{AO} = \frac{x + 1760}{x} = \cot 22^\circ = 2.475$; $\therefore 1.475x = 1760$.

$$\therefore \text{height of hill} = x = 1193 \text{ yds. nearly.}$$

18. See figure on page 44.

Let OA represent the lighthouse, B, C the two rocks.

Then $OA = 80$ yds., $\angle OBA = 75^\circ$, $\angle OCB = 15^\circ$, $\angle COA = 75^\circ$;

$$\therefore AB = OA \cot 75^\circ = 80 \times 0.268 \text{ yds.}$$

Let $CB = x$ yds.

Then $x + 80 \times 0.268 = OA \cot 15^\circ = 80 \times 3.732$, $\therefore x = 80 \times 3.464 = 277.12$ yds.

$$\therefore \text{required distance} = 277.12 \text{ yds.}$$

EXAMPLES. VI b. PAGE 47.

1. Let A, B be the two positions of the observer, P, Q the two objects.

Then $AB = 800$ yds., and PQA is a straight line making $\angle PAB$ equal to 45° .

Also $\angle PBA = 90^\circ$, $\angle QBA = 45^\circ$. $\therefore QA = QB = QP$.

And $QA = AB \cos 45^\circ = \frac{800}{\sqrt{2}} = 565.6$ yds. $PA = 2QA = 1131.2$ yds.

Thus the required distances are 565.6 yds., 1131.2 yds.

2. Let A, B be the two positions of the observer, P, Q the two ships; then APQ is a straight line at right angles to AB .

And $AB = 3$ miles, $\angle ABP = 30^\circ$, $\angle ABQ = 60^\circ$.

$\therefore BP = AB \sec 30^\circ = \frac{6}{\sqrt{3}} = 3.464$ miles, $BQ = AB \sec 60^\circ = 6$ miles.

Thus the required distances are 3.464 miles, 6 miles.

3. Let O represent the harbour and ON, OE, OS, OW the directions of North, South, East, West.

Let P, Q be the positions of the two ships at 2 p.m.

$$\text{Then } \angle POW = 28^\circ, \quad \angle QOE = 62^\circ; \quad \therefore \angle POQ = 90^\circ.$$

$$\text{Also } OP = 2 \times 10 = 20 \text{ miles.} \quad OQ = 2 \times 10\frac{1}{2} = 21 \text{ miles.}$$

$$\therefore \text{distance} = PQ = \sqrt{20^2 + 21^2} = 29 \text{ miles.}$$

4. Let O be the position of the lighthouse, and P, Q the points at which the steamer enters and leaves the light.

Then PQ lies East and West, and OP, OQ are the directions of N.E., N.W.

$$\therefore \angle POQ = 90^\circ, \quad \angle PQO = \angle QPO = 45^\circ, \quad OP = OQ = 5 \text{ miles.}$$

$$\therefore PQ = \sqrt{25 + 25} = 5\sqrt{2} \text{ miles.} \quad \therefore \text{steamer sails } 5\sqrt{2} \text{ miles in } 30\sqrt{2} \text{ minutes,}$$

that is, the speed of steamer is 10 miles per hour.

5. Let O, P, Q be the first positions of the ship and lighthouses. OA the direction in which the ship is sailing, A its second position.

$$\text{Then } OA = 10 \text{ miles,} \quad \angle OAP = 45^\circ, \quad \angle AOP = 90^\circ, \quad \angle PAQ = 22\frac{1}{2}^\circ.$$

$$\therefore \angle PQA = 45^\circ - 22\frac{1}{2}^\circ = 22\frac{1}{2}^\circ = PAQ; \quad \therefore PA = PQ.$$

$$\text{And } OP = OA = 10 \text{ miles,} \quad PA = OA \sec 45^\circ = 10\sqrt{2} \text{ miles.}$$

$$\therefore OQ = OP + PQ = OP + PA = 10(\sqrt{2} + 1) = 24.14 \text{ miles.}$$

\therefore distances are 10 miles, 24.14 miles.

6. As before let O be the port, and ON, OE, OS, OW the directions of the cardinal points of the compass.

Let P, Q be the positions of the ships at the end of an hour.

$$\text{Then } OP = 8 \text{ miles,} \quad OQ = 8\sqrt{3} \text{ miles.}$$

$$\text{And } \angle PON = 35^\circ, \quad \angle QOS = 55^\circ; \quad \therefore \angle POQ = 90^\circ.$$

$$\therefore \text{distances apart} = PQ = \sqrt{64 + 3 \times 64} = 16 \text{ miles.}$$

$$\text{Also } \tan QPO = \frac{8\sqrt{3}}{8} = \sqrt{3}; \quad \therefore \angle QPO = 60^\circ.$$

$$\therefore \angle QPO - \angle PON = 60^\circ - 35^\circ = 25^\circ;$$

\therefore bearing of the second vessel as observed from the first is S. 25° W.

7. Let A be the lighthouse, O, P the two positions of the vessel.

Then AP the direction of S., AO the direction of E.S.E., OP the direction of S.S.W.

$$\therefore \angle AOP = 90^\circ; \quad \angle PAO = 90^\circ - 22\frac{1}{2}^\circ = 67\frac{1}{2}^\circ; \quad \text{and } AO = 4 \text{ miles.}$$

$$\therefore PO = AO \tan 67\frac{1}{2}^\circ = 4 \times 2.414 = 9.656 \text{ miles.}$$

\therefore the vessel sails at the rate of 9.656 miles per hour.

8. We have $\angle CAB = 10^\circ + 50^\circ = 60^\circ$, $\angle ABC = 180^\circ - 50^\circ - 40^\circ = 90^\circ$,

$$BC = 10 \text{ miles.}$$

$$\therefore AB = BC \cot 60^\circ = \frac{10}{\sqrt{3}} = 5\cdot77 \text{ mls.}; AC = BC \cosec 60 = \frac{20}{\sqrt{3}} = 11\cdot54 \text{ mls.}$$

9. Let O be the lighthouse, and A , B the two positions of the ship.

Then $\angle OAB = \angle OBA = 45^\circ$; $OA = OB = 15$ miles.

$$\therefore AB = 15\sqrt{2} \text{ miles} = \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{the ship in } 1\frac{1}{2} \text{ hours sails } \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{in a day it sails } \frac{15\sqrt{2} \times 60}{69} \times \frac{24 \times 2}{3} \text{ knots, that is, } 295\cdot09 \text{ knots.}$$

10. Let O be the lighthouse, A , B the two positions of the coaster.

Then AB is in direction S.E., and OA is in direction N.E.; $\therefore \angle OAB = 90^\circ$.

Also $\angle AOB = 45^\circ + 15^\circ = 60^\circ$, and $OA = 9$ miles;

$\therefore AB = OA \tan 60^\circ = 9\sqrt{3}$ miles; \therefore coaster sails $9\sqrt{3}$ miles in 3 hours.

\therefore rate of the coaster's sailing = 5.196 miles per hour.

Also $OB = AO \sec 60^\circ = 18$ miles.

That is, the distance of the coaster from the lighthouse at time of second observation = 18 miles.

11. Let P be the position of the vessel when it is N.E. of A and N.W. of B . Then $\angle APB = 90^\circ$.

Also $\angle PAB = 45^\circ - 15^\circ = 30^\circ$. $\therefore PA = AB \cos 30^\circ = 6\sqrt{3}$ miles.

Now the direction S. 15° E. is at right angles to the direction E. 15° N.

\therefore the ship crosses AB at right angles. Draw PN perpendicular to AB .

Then $PN = AP \sin 30^\circ = 3\sqrt{3}$ miles; therefore the ship will reach N in $\frac{3\sqrt{3}}{10}$ hours, that is in 31.176 minutes.

\therefore the ship will cross the line at about 31' past midnight.

12. Let P , Q be the two spires.

Then $\angle PAB = 90^\circ$, and $\angle PBQ = 37\frac{1}{2}^\circ - 7\frac{1}{2}^\circ = 30^\circ$;

$$\therefore \angle QBA = 90^\circ - 37\frac{1}{2}^\circ - 22\frac{1}{2}^\circ = 30^\circ;$$

$\therefore \angle BPQ = 30^\circ = \angle PBQ$; so that $QB = QP = 1\cdot5$ miles.

$$\therefore AB = BQ \cos 30^\circ = \frac{3\sqrt{3}}{4} \text{ miles.}$$

\therefore the train travels $\frac{3\sqrt{3}}{4}$ miles in 2 minutes,

that is, $\frac{3\sqrt{3}}{4} \times 30$ miles per hour, or 38.97 miles per hour.

EXAMPLES. VI. c. PAGE 48 A.

1. $h = 83 \tan 23^\circ 44' = 83 \times .4397$ yards
 $= 109$ ft., approximately.

2. $h = 173 \tan 63^\circ = 173 \times 1.9626$ ft.
 $= 339.53$ ft.

3. $h = 200 \sin 54^\circ = 200 \times .8090$ metres
 $= 161.8$ metres.

4. $d = 500 \sin 23^\circ = 500 \times 3 \times .3907$ ft.
 $= 586.05$ ft.

5. The distance between two consecutive posts = $\frac{1760}{22} = 80$ yds.

Then required distance = $80 \tan 16^\circ 42' = 80 \times .3000$ yds.
 $= 24$ yds.

6. Let $ABCD$ be the square, and let the line be drawn from B to E , the middle point of AD .

Then $\tan ABE = \frac{AE}{AB} = .5$, whence $\angle ABE = 26^\circ 34'$;
 $\therefore \angle EBC = 90^\circ - 26^\circ 34' = 63^\circ 26'$.

7. Let D be the middle point of the base BC of the isosceles $\triangle ABC$, in which $AB = 3BC$.

Then $\cos DBA = \frac{BD}{BA} = \frac{1}{6} = .1666$,

whence $\angle B = 80^\circ 25' = \angle C$; $\therefore A = 19^\circ 10'$.

8. With the figure on p. 41, $QR = 160$ ft., $\angle PRT = 45^\circ$, $\angle PQT = 21^\circ 48'$; also $PT = RT$. If h is the required height in feet

$$\frac{h}{h+160} = \tan 21^\circ 48' = .4000;$$

$$\therefore h = .4h + 64; \therefore .6h = 64, \text{ and } h = 107.$$

9. With the same figure as in Ex. 8, $QR = 100$ yds., $\angle PRT = 54^\circ 24'$, $\angle PQT = 27^\circ 12'$. Let h be the height in feet; then

$$\frac{300 + RT}{h} = \cot 27^\circ 12' = \tan 62^\circ 48' = 1.9458;$$

and $RT = h \cot 54^\circ 24' = h \tan 35^\circ 36' = h \times .7159$;
 $\therefore 300 + h \times .7159 = h \times 1.9458$;

that is, $1.2299h = 300$; whence $h = 244$, nearly.

Or thus: Since $\angle RPQ = 27^\circ 12'$, $\therefore PR = QR = 100$ yds;
 $\therefore h = 300 \sin 54^\circ 24' = 300 \times .8131 = 244$.

10. With the same figure and notation as in Ex. 9,

$$\frac{1760+RT}{h} = \tan 73^\circ 18' = 3.3332;$$

and

$$RT = h \tan 35^\circ = h \times 7002;$$

$$\therefore 1760 + h \times 7002 = h \times 3.3332;$$

that is, $2.6330h = 1760$; whence $h = 668$, nearly.

11. Let AB represent the top, and DE the bottom of the trench. Draw AC and BF perp. to ED and DE produced.

$$\text{Then } CD = 8 \tan 12^\circ = 8 \times 2.126 = 17.008 \text{ ft.};$$

$$\text{whence } CE = 10.7008 \text{ ft. } \therefore EF = 4.2992 \text{ ft.}$$

$$\text{In } \triangle EBF, \quad \tan B = \frac{4.2992}{8} = .5374;$$

$$\text{whence } B = 28^\circ 15'.$$

12. Let C and B be the first and second positions of the observer; then $\angle ACB = 90^\circ$, and $\angle ADB = 143^\circ 24'$. $\therefore \angle ADC = 36^\circ 36'$.

$$\text{Now } AC = 630 \tan 36^\circ 36' = 630 \times 7.427 = 467.9 \text{ m.}$$

$$AD = \frac{63}{\cos 36^\circ 36'} = \frac{63}{.8028} = 784.7 \text{ m.}$$

13. Let A be the point of observation, B the top, and C the bottom of the tower. Draw AE horizontally to meet BC in E . Then $\angle EAC = 17^\circ$, and $EC = AD = 30$ ft.

$$\begin{aligned} AE &= EC \cot 17^\circ = 30 \tan 73^\circ = (30 \times 3.2709) \text{ ft.} \\ &= 98.127 \text{ ft.} \end{aligned}$$

$$\begin{aligned} \text{Again } BE &= AE \tan 42^\circ = (98.127 \times 9.004) \text{ ft.} \\ &= 88.3506 \text{ ft.}; \\ \therefore \text{ required height} &= (30 + 88.3506) \text{ ft.} \\ &= 118.35 \text{ ft.} \end{aligned}$$

14. See fig. on page 44. Let $OA = x$, $AB = y$;

$$\text{then } y = x \cot 35^\circ = x \tan 55^\circ = x \times 1.4281,$$

$$\frac{700+y}{x} = \cot 14^\circ = \tan 76^\circ = 4.0108;$$

$$\therefore 700 + x \times 1.4281 = x \times 4.0108;$$

$$\text{that is, } 700 = x \times 2.5827; \text{ whence } x = 271.$$

15. Let O be the lighthouse, and A , B the two positions of the ship. Then $\angle BOA = 90^\circ$, $\angle OBA = 32^\circ$, $AB = 15$ mi.

$$\begin{aligned} \text{Now } OA &= AB \sin ABO = 15 \sin 32^\circ \\ &= 15 \times 5299 = 7.9485 \text{ mi.} \end{aligned}$$

16. Let O be the house, and A, B the two positions. Then $\angle BOA = 90^\circ$, $\angle OBA = 52^\circ$, $AB = 2$ km. Let OC be perp. to AB ;

then

$$OB = 2000 \cos 52^\circ = 2000 \times 0.6157 = 1231.4 \text{ m.}$$

$$OC = OB \sin 52^\circ = 1231.4 \times 0.7880 = 970.3 \text{ m.}$$

17. Let L be the lighthouse, and S_1, S_2 the two positions of the ship. Then $\angle S_1OS_2 = 90^\circ$, $\angle S_2S_1L = 56^\circ$, $LS_1 = 12$ mi.

Now $S_1S_2 = \frac{12}{\cos 56^\circ} = \frac{12}{0.5592}$ mi.; and since the ship has been sailing $\frac{7}{6}$ of an hour, the number of miles per day $= \frac{12}{0.5592} \times \frac{6}{7} \times 24 = 441.5$.

18. Let B be the battery, and S_1, S_2 the two positions of the ship. Draw S_1C perp. to BS_2 ; then $\angle CS_1B = 45^\circ$, $S_1B = 2.5$ mi., and $BS_2 = 4$ mi.

$$\text{Now } S_1C = 2.5 \sin 45^\circ = 2.5 \times 0.7071 = 1.7678 \text{ mi.}$$

$$S_2C = 4 - 1.7678 = 2.2322 \text{ mi.}$$

$$\tan S_1S_2C = \frac{1.7678}{2.2322} = 0.7912; \text{ whence } \angle S_1S_2C = 38^\circ 23'.$$

$\therefore S_2$ lies $38^\circ 23'$ E. of N. from S_1 .

19. See fig. on page 47.

$$\text{Here } \angle BAE = 49^\circ, \angle EAC = 41^\circ; \therefore \angle BAC = 90^\circ.$$

$$\text{Also } \angle ACN' = 90^\circ - 41^\circ = 49^\circ, \text{ and } \angle BCN' = 15^\circ;$$

$$\therefore \angle ACB = 49^\circ - 15^\circ = 34^\circ.$$

$$\text{Now } AB = AC \tan 34^\circ = 20 \times 0.6745 = 13.49 \text{ mi.}$$

$$BC = \frac{AC}{\cos 34^\circ} = \frac{20}{0.8290} = 24.12 \text{ mi.}$$

EXAMPLES. VII. a. PAGE 54.

For Examples 1—22, see Art. 64; the following solutions will suffice as illustrations.

$$6. \text{ Radian measure of } 57\frac{1}{2} \text{ degrees} = \frac{57\frac{1}{2}}{180} \pi = \frac{23\pi}{72}.$$

$$7. \text{ Radian measure of } 14\frac{2}{5} \text{ degrees} = \frac{14\frac{2}{5}}{180} \pi = \frac{2\pi}{25}.$$

$$10. \text{ Radian measure of } 37\frac{1}{2} \text{ degrees} = \frac{37\frac{1}{2}}{180} \times 3.1416 = 0.6545.$$

$$11. \text{ Radian measure of } 68\frac{3}{4} \text{ degrees} = \frac{68\frac{3}{4}}{180} \times 3.1416 \\ = \frac{275}{4 \times 180} \times 3.1416 = \frac{55}{4 \times 3} \times 2618 = 1.1999.$$

$$16. \frac{7\pi}{45} \text{ radians} = \frac{7 \times 180}{45} \text{ degrees} = 28^\circ.$$

$$19. \cdot 3927 \text{ radians} = \cdot 3927 \times \frac{180^\circ}{\pi} [\text{Art. 63}] \\ = \frac{3927}{31416} \times 180^\circ = \frac{180^\circ}{8} = 22^\circ 30'.$$

$$22. 2.8798 \text{ radians} = 2.8798 \times \frac{180^\circ}{\pi} = \frac{28798}{31416} \times 180^\circ = \frac{11}{12} \times 180^\circ = 165^\circ.$$

$$23. \text{ Here } \frac{\theta}{\pi} = \frac{36.54}{180} = \frac{2.03}{10}; \quad \begin{array}{r} 60) 24 \\ 60) 32.4 \\ \quad \quad \quad \cdot 54 \end{array} \\ \therefore \theta = 2.03 \times \frac{22}{7} = \cdot 638.$$

$$25. \text{ Here } \frac{\theta}{\pi} = \frac{116.046}{180} = \cdot 6447; \quad \begin{array}{r} 60) 45.6 \\ 60) 2.76 \\ \quad \quad \quad \cdot 46 \end{array} \\ \therefore \theta = \cdot 6447 \times \frac{22}{7} = 2.0262.$$

$$27. \text{ A radian} = \frac{180}{\pi} \text{ degrees;} \\ \therefore \text{no. of seconds in a radian} = \frac{180 \times 60 \times 60}{\pi} \\ = 180 \times 60 \times 60 \times \cdot 31831 \\ = 206265 \text{ nearly.}$$

$$28. \text{ Since } 1^\circ = \frac{\pi}{180} \text{ radians, the radian measure of } 1'' \\ = \frac{3.1416}{180 \times 60 \times 60} = \cdot 0000048.$$

EXAMPLES. VII. b. PAGE 56.

$$5. \cot^2 \frac{\pi}{6} + 4 \cos^2 \frac{\pi}{4} + 3 \sec^2 \frac{\pi}{6} = (\sqrt{3})^2 + 4 \left(\frac{1}{\sqrt{2}} \right)^2 + 3 \left(\frac{2}{\sqrt{3}} \right)^2 = 3 + 2 + 4 = 9.$$

$$6. 3 \tan^2 \frac{\pi}{6} - \frac{1}{3} \sin^2 \frac{\pi}{3} - \frac{1}{2} \operatorname{cosec}^2 \frac{\pi}{4} + \frac{4}{3} \cos^2 \frac{\pi}{6} \\ = 3 \left(\frac{1}{\sqrt{3}} \right)^2 - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (2)^2 + \frac{4}{3} \left(\frac{\sqrt{3}}{2} \right)^2 \\ = 1 - \frac{1}{4} - 1 + 1 = \frac{3}{4}.$$

$$7. \left(\sin \frac{\pi}{6} + \cos \frac{\pi}{6} \right) \left(\sin \frac{\pi}{3} - \cos \frac{\pi}{3} \right) \sec \frac{\pi}{3}$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) 2 = 2 \left(\frac{3}{4} - \frac{1}{4} \right) = 1.$$

8. First side = $\sin \theta \operatorname{cosec} \theta - \cot \theta \tan \theta = 1 - 1 = 0$.

9. First side = $\frac{\cos^2 \theta}{\sin \theta} - \cot^2 \theta \sin \theta = \frac{\cos^2 \theta}{\sin \theta} - \frac{\cos^2 \theta}{\sin \theta} = 0$.

10. First side = $\frac{\cos^2 \theta}{\operatorname{cosec} \theta} \cdot \frac{\sec \theta}{\tan \theta} = \cos^2 \theta \sin \theta \times \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} = \cos^2 \theta$.

11. First side = $\tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$

$$= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = \sec \theta \operatorname{cosec} \theta = \sec \theta \sec \left(\frac{\pi}{2} - \theta \right).$$

12. First side = $\sec^2 \theta + \operatorname{cosec}^2 \theta = \sec^2 \theta (1 + \cot^2 \theta)$

$$= \sec^2 \theta \operatorname{cosec}^2 \theta = (1 + \tan^2 \theta) \sec^2 \left(\frac{\pi}{2} - \theta \right).$$

14. See Art. 69.

15. The second side = $\frac{1}{\cos^2 \frac{\pi}{3}} - \frac{1}{\cos^2 \frac{\pi}{6}} = 4 - \frac{4}{3} = \frac{8}{3}$

$$= 3 - \frac{1}{3} = \tan^2 \frac{\pi}{3} - \cot^2 \frac{\pi}{3}.$$

16. The expression = $(\sin \theta + \cos \theta)^2 + (\cos \theta - \sin \theta)^2$

$$= 2(\sin^2 \theta + \cos^2 \theta) = 2.$$

EXAMPLES. VII. c. PAGE 60.

1. Here $\frac{\text{arc}}{\text{radius}} = \frac{1.6}{8} = \frac{1}{5}$ = radian measure required.

2. Here $r = \frac{a}{\theta} = \frac{219}{7.3} = 300$ ft. 3. Radian measure = $\frac{7.5}{7.5} = 1$.

4. Here $a = r\theta = 1.625 \times 3.6 = 5.85$ yds.

5. Here $a = 627$ inches; $\theta = 1.9$. $\therefore r = \frac{a}{\theta} = \frac{627}{1.9} = 330$ inches.

6. Each revolution = 2π radians;

$$\therefore 5 \text{ radians} = \frac{5}{2\pi} = \frac{7 \times 5}{44} \text{ revolutions.}$$

And each revolution takes $\frac{1}{35}$ of a second;

$$\therefore \text{required time} = \frac{1}{35} \times \frac{35}{44} = \frac{1}{44} \text{ of a second.}$$

7. Here $a=r\theta$, where $r=28$ inches and $\theta=\frac{1}{3} \times \frac{44}{7}$.

$$\therefore a = \frac{44}{21} \times 28 = 58\frac{2}{3} \text{ inches.}$$

8. Radian measure of $75^\circ = \frac{75 \times 3.1416}{180}$.

If r be the length of the rope in yards, we have

$$r = \frac{52.36 \times 180}{3.1416 \times 75} = 40 \text{ yards.}$$

9. Here $a=r\theta$, where $r=3960$ miles, and $\theta=\text{radian measure of } 1 \text{ minute}$;

$$\therefore a = 3960 \times \frac{3.1416}{180 \times 60} = 1.15192 \text{ miles.}$$

10. The number of radians in the angle $= \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3}$;

$$\therefore \text{number of seconds} = \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3} \times \frac{180}{3.1416} \times 60 \times 60 \\ = 17.904, \text{ on reduction.}$$

11. With the figure on p. 59, we have to find the angle POQ when $PO=3960$ miles, and $PQ=145.2$.

$$\therefore \text{radian measure} = \frac{145.2}{3960};$$

$$\therefore \text{no. of degrees} = \frac{145.2}{3960} \times \frac{180 \times 7}{22} = \frac{1452}{220} \times \frac{7}{22} = \frac{66 \times 7}{220} = \frac{21}{10}.$$

$$\therefore \text{the angle} = 2^\circ 6'.$$

12. Here $r = \frac{a}{\theta}$, where $a=1$ foot, and θ is the radian measure of $\frac{14}{11}$ degrees.

$$\therefore r = 1 \div \left(\frac{14}{11} \times \frac{\pi}{180} \right) = \frac{11}{14} \times \frac{180 \times 7}{22} = 45 \text{ ft.}$$

MISCELLANEOUS EXAMPLES. B. PAGE 61.

$$1. D = \frac{180}{\pi} \times 15708 = \frac{180 \times 15708}{314160} = 9.$$

$$2. \text{ See figure on p. 14. } b = c \cos A = \frac{110\sqrt{3}}{2} = 55 \times 1.732 = 95.26.$$

3. If $\frac{12\pi}{23}$ is represented by $\frac{8}{5}$,

$$\pi \dots \dots \dots \frac{8}{5} \times \frac{25}{12}, \text{ or } \frac{10}{3}.$$

$\therefore 180 \div \frac{10}{3}$ is the number of degrees in the unit angle; that is, the unit is 54° .

4. Here $r = \frac{a}{\theta}$, where $a = 1$ inch and θ is the radian measure of $1'$.

$$\therefore r = 1 \div \left(\frac{\pi}{180} \times \frac{1}{60} \right) = 180 \times 60 \times \frac{1}{\pi} = 180 \times 60 \times \frac{1}{31831} = 3438 \text{ inches.}$$

$$\begin{aligned} 5. (1) \quad \text{First side} &= (\sin \alpha + \cos \alpha) \left(\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha} \right) \\ &= \frac{(\sin \alpha + \cos \alpha)(\sin^2 \alpha + \cos^2 \alpha)}{\sin \alpha \cos \alpha} = \sec \alpha + \operatorname{cosec} \alpha. \end{aligned}$$

$$\begin{aligned} (2) \quad \text{First side} &= (\sqrt{3} + 1)(3 - \sqrt{3}) = \sqrt{3}(\sqrt{3} + 1)(\sqrt{3} - 1) \\ &= 3\sqrt{3} - \sqrt{3} = \tan^3 60^\circ - 2 \sin 60^\circ. \end{aligned}$$

6. In the figure on p. 14, if BC represents the chimney and AC the shadow we have

$$\tan A = \frac{60}{3 \times 20\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}; \quad \therefore A = 30^\circ.$$

$$\begin{aligned} 7. (1) \quad \text{First side} &= 2 \tan^2 \theta + 5 \tan \theta + 2 \\ &= 2(1 + \tan^2 \theta) + 5 \tan \theta = 2 \sec^2 \theta + 5 \tan \theta. \end{aligned}$$

$$(2) \quad \frac{\cot^2 \alpha}{1 + \operatorname{cosec} \alpha} = \frac{\operatorname{cosec}^2 \alpha - 1}{\operatorname{cosec} \alpha + 1} = \operatorname{cosec} \alpha - 1.$$

8. Expressed in radians the third angle $= \pi - \left(\frac{\pi}{4} + \frac{5\pi}{8} \right) = \frac{\pi}{8}$. The sexagesimal equivalent is $22\frac{1}{2}^\circ$.

9. Let x be the number of degrees in the angle; then

$$x = 14 \left(\frac{\pi}{180} x \right) + 51, \text{ or } x - \frac{11}{45} x = 51; \text{ whence } x = 67\frac{1}{2}.$$

10. With the figure of Art. 45, we have

$$a = b \tan 60^\circ = 6\sqrt{3}; \quad c = b \sec 60^\circ = 6 \times 2 = 12.$$

Also the perpendicular from C on $AB = b \sin 60^\circ = 3\sqrt{3}$.

11. (1) First side = $\cot \theta + \tan \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$
 $= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta = \operatorname{cosec} \theta \operatorname{cosec} \left(\frac{\pi}{2} - \theta \right).$
- (2) First side = $\operatorname{cosec}^2 \theta + \sec^2 \theta = \sec^2 \theta \operatorname{cosec}^2 \theta$ [Art. 31, Ex. 1.]
 $= \operatorname{cosec}^2 \theta \operatorname{cosec}^2 \left(\frac{\pi}{2} - \theta \right).$

12. In the figure on p. 41, let PT be the pillar; then
 $QR=20$ ft., $\angle PQR=30^\circ$, $\angle PRT=60^\circ$, and $PR=QR=20$ ft.

$$\therefore PT=PR \sin 60^\circ = \frac{20\sqrt{3}}{2} = 10\sqrt{3}=17.32 \text{ ft.}$$

13. First side = $\sin^2 A \left(\frac{1}{\cos^2 A} - 1 \right) = \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A}$
 $= \sin^2 A \cdot \sin^2 A \cdot \sec^2 A = \sin^4 A \sec^2 A.$

14. x grades = $\frac{9x}{10}$ degrees;
also $\frac{\pi x}{300}$ radians = $\frac{\pi x}{300} \times \frac{180}{\pi} = \frac{3x}{5}$ degrees;
 $\therefore 3x + \frac{9x}{10} + \frac{3x}{5} = 180$; whence $x=40$. Thus the angles are 120° , 36° , 24° .

15. Expression = $\left(\frac{\sqrt{3}}{2} \right)^3 \sqrt{3} - 2(\sqrt{2})^2 + 3 \cdot \frac{1}{2} \cdot 1 - (\sqrt{3})^2$
 $= \frac{9}{8} - 4 + \frac{3}{2} - 3 = -\frac{35}{8}.$

16. (1) First side = $1 + \tan^2 A + 2 \tan A + 1 + \cot^2 A + 2 \cot A$
 $= \sec^2 A + \operatorname{cosec}^2 A + 2 \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$
 $= \sec^2 A + \operatorname{cosec}^2 A + 2 \sec A \operatorname{cosec} A = (\sec A + \operatorname{cosec} A)^2.$
- (2) First side = $(\sec a - 1)^2 - \sin^2 a (\sec a - 1)^2 = (\sec a - 1)^2 (1 - \sin^2 a)$
 $= (\sec a - 1)^2 \cos^2 a = (1 - \cos a)^2.$

17. (1) $a^2 + b^2 > 2ab$, since $(a-b)^2$ is positive.
Therefore $\frac{a^2 + b^2}{2ab} > 1$. Hence $\operatorname{cosec} \theta = \frac{a^2 + b^2}{2ab}$ is possible.
- (2) $a^2 + 1 > 2a$: so that $a + \frac{1}{a} > 2$.

Hence $2 \sin \theta = a + \frac{1}{a}$ is impossible [Art. 16].

18. The height of the balloon = $660 \tan 60^\circ$ ft. = $660\sqrt{3}$ feet.

\therefore the balloon rises $660\sqrt{3}$ feet in 1.5 minutes.

\therefore it rises $\frac{660\sqrt{3} \times 60}{1.5 \times 3 \times 1760}$, or 8.66 miles per hour.

19. Let x° be the common difference between the angles, then they are $36^\circ, 36^\circ + x^\circ, 36^\circ + 2x^\circ$,

$$\therefore 3x + 3 \times 36 = 180; \text{ whence } x = 24.$$

\therefore the angles are $36^\circ, 60^\circ, 84^\circ$, or $\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}$ radians.

20. First side = $\sin^2 \alpha (1 + \tan^2 \beta) + \tan^2 \beta (1 - \sin^2 \alpha) = \sin^2 \alpha + \tan^2 \beta$.

21. Let $CD = x$ = the perpendicular.

Then $\angle CBD = 180^\circ - 116^\circ 33' = 63^\circ 27'$; $\therefore x = DB \tan 63^\circ 27' = 2DB$.

And $x = DA \tan 42^\circ = \left(\frac{x}{2} + 55\right) \times 9$; whence $x = 4.5 \times 20 = 90$.

22. (1) First side = $\frac{\cos \alpha (1 + \cos \alpha) + \sin^2 \alpha}{\sin \alpha (1 + \cos \alpha)} = \frac{\cos \alpha + 1}{\sin \alpha (1 + \cos \alpha)} = \operatorname{cosec} \alpha$.

(2) First side = $\frac{1 - \cos \alpha}{\sin \alpha} \left(\frac{1}{\cos \alpha} - \cos \alpha \right) = (1 - \cos \alpha) \frac{\sin \alpha}{\cos \alpha} = \tan \alpha - \sin \alpha$.

23. First side = $\left(\frac{1 + \sqrt{3}}{1 - \sqrt{3}}\right)^2 = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{1 + \cos 30^\circ}{1 - \cos 30^\circ}$.

24. Let the man start from A and walk to B , and let C denote the position of the windmill.

Then we have $\angle ACB = 90^\circ$, $\angle CAB = 30^\circ$, $BC = 1$ mile.

$\therefore AB = BC \operatorname{cosec} 30^\circ = 2$ miles. $AC = BC \tan 60^\circ = 1.732$ miles.

And rate of walking is 2 miles per half hour, or 4 miles an hour.

25. The complement of $\frac{3\pi}{8} = \frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8}$ radians.

26. (1) $3 \sin \theta + 4 - 4 \sin^2 \theta = \frac{9}{2}$, (2) $\tan \theta + \frac{2}{\sqrt{3}} = \frac{1}{\tan \theta}$,

$$8 \sin^2 \theta - 6 \sin \theta + 1 = 0; \quad \sqrt{3} \tan^2 \theta + 2 \tan \theta - \sqrt{3} = 0;$$

$$\therefore (4 \sin \theta - 1)(2 \sin \theta - 1) = 0; \quad \therefore (\sqrt{3} \tan \theta - 1)(\tan \theta + \sqrt{3}) = 0;$$

$$\therefore \sin \theta = \frac{1}{4}, \text{ or } \frac{1}{2}, \quad \therefore \tan \theta = \frac{1}{\sqrt{3}}, \text{ or } -\sqrt{3}.$$

$$\therefore \theta = 30^\circ, \text{ or } 14^\circ 29'. \quad \therefore \theta = 30^\circ.$$

27. Here $\frac{5 \sin \alpha - 3 \cos \alpha}{\sin \alpha + 2 \cos \alpha} = \frac{5 \tan \alpha - 3}{\tan \alpha + 2} = \frac{20 - 15}{4 + 10} = \frac{5}{14}.$

28. First side $= \frac{1 - \sin A \cos A}{\cos A \cdot \frac{\sin A - \cos A}{\sin A \cos A}} \times \frac{\sin A - \cos A}{\sin^2 A - \sin A \cos A + \cos^2 A}$
 $= \frac{\sin A (1 - \sin A \cos A)}{1 - \sin A \cos A} = \sin A.$

29. The distance $= 195.2 \operatorname{cosec} 77^\circ 26' \text{ yds.} = \frac{195.2}{.976} = 200 \text{ yds.}$

30. We have $70^\circ = \frac{7\pi}{18}$ radians.

\therefore distance required $= 27 \times \frac{7\pi}{18} \text{ feet} = \frac{21}{2} \times \frac{22}{7} = 33 \text{ feet.}$

EXAMPLES. VIII. a. PAGE 70.

18. $\sin 420^\circ = \sin (360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$

20. $\tan (-315^\circ) = \tan (-360^\circ + 45^\circ) = \tan 45^\circ = 1.$

22. $\operatorname{cosec} (-330^\circ) = \operatorname{cosec} (-360^\circ + 30^\circ) = \operatorname{cosec} 30^\circ = 2.$

24. $\cot \frac{17\pi}{4} = \cot \left(4\pi + \frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1.$

26. $\tan \left(-\frac{5\pi}{3}\right) = \tan \left(-2\pi + \frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}.$

EXAMPLES. VIII. b. PAGE 72.

1. The boundary line of 120° lies in the second quadrant,

$$\therefore \cos 120^\circ = -\sqrt{1 - \sin^2 120^\circ} = -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2};$$

$$\therefore \tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = -\sqrt{3}.$$

2. The boundary line of 135° lies in the second quadrant;

$$\therefore \sec 135^\circ = -\sqrt{1 + \tan^2 135^\circ} = -\sqrt{2};$$

and $\sin 135^\circ \sec 135^\circ = -1; \quad \therefore \sin 135^\circ = \frac{1}{\sqrt{2}}.$

3. The boundary line of 240° lies in the third quadrant;

$$\therefore \sec 240^\circ = -\sqrt{1 + \tan^2 240^\circ} = -2; \quad \therefore \cos 240^\circ = -\frac{1}{2}.$$

4. The boundary line of $202^\circ 37'$ lies in the third quadrant;

$$\therefore \cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{12}{5}.$$

5. The boundary line of $143^\circ 8'$ lies in the second quadrant; and

$$\operatorname{cosec} A = 1\frac{2}{5}; \quad \therefore \sin A = \frac{3}{5}; \quad \therefore \cos A = -\sqrt{1 - \sin^2 A} = -\frac{4}{5};$$

$$\therefore \sec A = -\frac{5}{4}. \quad \text{Hence } \tan A = \frac{\sin A}{\cos A} = -\frac{3}{4}.$$

6. The boundary of $216^\circ 52'$ lies in the third quadrant;

$$\therefore \sin A = -\sqrt{1 - \cos^2 A} = -\sqrt{1 - \frac{16}{25}} = -\frac{3}{5}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{4}{3}.$$

7. The boundary line of $\frac{2\pi}{3}$ lies in the second quadrant;

$$\text{and } \sec \frac{2\pi}{3} = -2; \quad \therefore \cos \frac{2\pi}{3} = -\frac{1}{2}. \quad \therefore \sin \frac{2\pi}{3} = +\sqrt{1 - \cos^2 \frac{2\pi}{3}} = \frac{\sqrt{3}}{2}.$$

$$\cot \frac{2\pi}{3} = \frac{\cos \frac{2\pi}{3}}{\sin \frac{2\pi}{3}} = -\frac{1}{\sqrt{3}}.$$

8. The boundary line of $\frac{5\pi}{4}$ lies in the third quadrant;

$$\therefore \cos \frac{5\pi}{4} = -\sqrt{1 - \sin^2 \frac{5\pi}{4}} = -\frac{1}{\sqrt{2}};$$

$$\therefore \sec \frac{5\pi}{4} = -\sqrt{2}, \quad \text{and } \tan \frac{5\pi}{4} = \frac{\sin \frac{5\pi}{4}}{\cos \frac{5\pi}{4}} = 1.$$

9. We have $\sin A = \pm \sqrt{1 - \cos^2 A} = \pm \sqrt{1 - \frac{144}{169}} = \pm \frac{5}{13}.$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \pm \frac{5}{12}.$$

EXAMPLES. IX. PAGE 79.

6. Expression = $1 \cdot (-1)^2 - 2(-1) \cdot 1 = 1 + 2 = 3$.
7. Expression = $3 \cdot 0 \cdot (-1) + 2 \cdot 1 - 1 = 2 - 1 = 1$.
8. Expression = $2 \cdot (-1)^2 \cdot 1 + 3(-1)^3 - 1 = 2 - 3 - 1 = -2$.
9. Expression = $0 \times 0 + 1 - (-1) = 1 + 1 = 2$.

MISCELLANEOUS EXAMPLES. C. PAGE 80.

1. If $\tan A = -\frac{3}{4}$, the boundary of A will lie either in second or fourth quadrant. [See figure on page 72.]

In either position the radius vector = $\sqrt{3^2 + 4^2} = 5$.

$$\text{Hence } \cos XOP = -\frac{4}{5}; \quad \cos XOP' = \frac{4}{5}.$$

2. First side = $(2 + \sin A)(1 - 2 \sin A) \sec A$
 $= (2 - 3 \sin A - 2 \sin^2 A) \sec A$
 $= (2 \cos^2 A - 3 \sin A) \sec A = 2 \cos A - 3 \tan A$.

3. $a = \sqrt{c^2 - b^2} = \sqrt{(21)^2 - (10\cdot 5)^2} = 21 \sqrt{1 - \frac{1}{4}} = \frac{21\sqrt{3}}{2}$.
 $\sin A = \frac{a}{c} = \frac{\sqrt{3}}{2}; \quad \text{whence } A = 60^\circ, B = 30^\circ$.

4. A lies between 180° and 270° ;
 $\therefore \tan A = +\sqrt{\sec^2 A - 1} = \sqrt{\left(\frac{25}{7}\right)^2 - 1} = \frac{24}{7}; \quad \text{and } \cot A = \frac{7}{24}$.

5. We have $19^\circ = \frac{19\pi}{180}$ radians.

Also the radius of the earth = 3960 miles.

$$\therefore \text{required distance} = \frac{19\pi}{180} \times 3960 = 19 \times 22 \times \pi = 1313 \text{ miles nearly.}$$

6. Let AB represent the cliff and P, Q the positions of the two boats.
 Then $AB = 200$ ft., $\angle APB = 34^\circ 30'$, $\angle AQB = 18^\circ 40'$,
 $\therefore QB = AB \cot 18^\circ 40' = 200 \times 2.96 = 592$ ft.
 $PB = AB \cot 34^\circ 30' = 200 \times 1.455 = 291$ ft.
 $\therefore \text{required distance} = QB - PB = 301$ ft.

7. Since the boundary line of A is in the third quadrant,

$$\therefore \sec A = -\sqrt{1+\tan^2 A} = -\sqrt{1+\frac{16}{9}} = -\frac{5}{3},$$

$$\therefore \cos A = -\frac{3}{5}, \text{ and } \sin A = -\frac{4}{5}.$$

$$\therefore 2 \cot A - 5 \cos A + \sin A = 2 \cdot \frac{3}{4} - 5 \left(-\frac{3}{5} \right) - \frac{4}{5} = \frac{3}{2} + 3 - \frac{4}{5} = 3 \frac{7}{10}.$$

8. We have $71^\circ 36' 3\cdot6'' = 71\cdot601^\circ = \frac{71\cdot601\pi}{180}$ radians.

$$\therefore \text{required radius} = 15 \div \frac{71\cdot601\pi}{180} = \frac{15 \times 180}{71\cdot601\pi} = 12\cdot003 \text{ inches.}$$

$$\begin{aligned} 9. \text{ First side} &= \frac{\tan^3 \theta}{\sec^2 \theta} + \frac{\cot^3 \theta}{\operatorname{cosec}^2 \theta} \\ &= \frac{\sin^3 \theta}{\cos \theta} + \frac{\cos^3 \theta}{\sin \theta} = \frac{\sin^4 \theta + \cos^4 \theta}{\sin \theta \cos \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta}. \end{aligned}$$

10. Let AB represent the flagstaff, BC the tower, and let D be the position of the observer.

Then $\angle BDC = 68^\circ 11'$, $\angle ADB = 2^\circ 10'$; $\therefore \angle ADC = 70^\circ 21'$.

Let $BC = x$ ft., then $x + 24 = DC \tan 70^\circ 21'$, and $DC = x \cot 68^\circ 11'$.

$\therefore x + 24 = x \cot 68^\circ 11' \tan 70^\circ 21'$; $\therefore x + 24 = x \times 2\cdot8 \times 4 = 1\cdot12x$;
whence $x = 200$; that is, the height of the tower is 200 ft.

11. If $\tan A = .5$, and $\tan B = .3333$, from the Tables we have $A = 26^\circ 34'$, $B = 18^\circ 26'$; $\therefore A + B = 45^\circ$.

12. We have $(4 \tan \theta - 3)(3 \tan \theta + 4) = 0$;

whence $\tan \theta = .75$, or -1.3333 .

\therefore , from the Tables, $\theta = 36^\circ 52'$, or $180^\circ - 53^\circ 8'$,
that is, $\theta = 36^\circ 52'$, or $126^\circ 52'$.

13. We have $\frac{3\cdot7}{r} = \text{radian measure of } 21^\circ 12'$

$$= \frac{\pi}{180} \times 21\cdot2;$$

$$\therefore 180 \times 3\cdot7 = r \times 3\cdot1416 \times 21\cdot2,$$

or

$$666 = r \times 66.602, \text{ approx.}$$

\therefore radius = 10 in., to the nearest inch.

If d be the number of miles between the two places

$$\frac{d}{4000} = \frac{3.7}{10}; \text{ whence } d = 1480.$$

14. Let T be the tower, and O_1, O_2 the two points of observation. Then it is easily seen that

$$\angle T O_1 O_2 = 30^\circ, \angle T O_2 O_1 = 60^\circ;$$

$$\therefore \angle O_1 T O_2 = 90^\circ; \text{ also } T O_2 = 2 \text{ km.}$$

$$\text{Now } O_1 T = O_2 T \tan 60^\circ = 2 \times 1.732 = 3.464 \text{ km.}$$

Also $O_1 O_2 = O_2 T \sec 60^\circ = 4 \text{ km.};$ and since he walks this distance in 40 min. his rate of walking is 6 km. per hour.

15. From the Tables, $\alpha + \beta = 51^\circ 33', \alpha - \beta = 47^\circ 5',$

$$\therefore \alpha = 49^\circ 19', \beta = 2^\circ 14'.$$

16. The expression $= \frac{10 - 6 \cot \alpha}{4 + 3 \cot \alpha} = \frac{10 - 4}{4 + 2} = 1.$

17. Let F be the fort, and S_1, S_2 the two positions of the ship. Then it is easily seen that $\angle F S_2 S_1 = 90^\circ, \angle S_2 S_1 F = 43^\circ;$ also $S_1 S_2 = 20 \text{ mi.}$

$$\therefore F S_2 = 20 \tan 43^\circ = 20 \times 0.9325 = 18.65 \text{ mi.}$$

$$F S_1 = \frac{20}{\cos 43^\circ} = \frac{20}{0.7314} = 27.346 \text{ mi.}$$

EXAMPLES. X. a. PAGE 87.

$$1. \cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}.$$

$$2. \sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$$

$$3. \tan 240^\circ = \tan (180^\circ + 60^\circ) = \tan 60^\circ = \sqrt{3}.$$

$$4. \operatorname{cosec} 225^\circ = \operatorname{cosec} (180^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$$

$$5. \sin (-120^\circ) = -\sin 120^\circ = -\sin (180^\circ - 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}.$$

$$6. \cot (-135^\circ) = -\cot 135^\circ = -\cot (180^\circ - 45^\circ) = \cot 45^\circ = 1.$$

$$7. \cot 315^\circ = \cot (180^\circ + 135^\circ) = \cot 135^\circ = -1.$$

$$8. \cos (-240^\circ) = \cos 240^\circ = \cos (180^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}.$$

9. $\sec(-300^\circ) = \sec 300^\circ = \sec(180^\circ + 120^\circ) = -\sec 120^\circ = -\sec(180^\circ - 60^\circ) = \sec 60^\circ = 2.$
10. $\tan \frac{3\pi}{4} = \tan \left(\pi - \frac{\pi}{4}\right) = -\tan \frac{\pi}{4} = -1.$
11. $\sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$
12. $\sec \frac{2\pi}{3} = \sec \left(\pi - \frac{\pi}{3}\right) = -\sec \frac{\pi}{3} = -2.$
13. $\operatorname{cosec} \left(-\frac{\pi}{6}\right) = -\operatorname{cosec} \frac{\pi}{6} = -2.$
14. $\cos \left(-\frac{3\pi}{4}\right) = \cos \frac{3\pi}{4} = \cos \left(\pi - \frac{\pi}{4}\right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
15. $\cot \left(-\frac{5\pi}{6}\right) = -\cot \frac{5\pi}{6} = -\cot \left(\pi - \frac{\pi}{6}\right) = \cot \frac{\pi}{6} = \sqrt{3}.$
16. $\cos(270^\circ + A) = \cos(180^\circ + 90^\circ + A) = -\cos(90^\circ + A) = \sin A.$
17. $\cot(270^\circ - A) = \cot(180^\circ + 90^\circ - A) = \cot(90^\circ - A) = \tan A.$
18. $\sin(A - 90^\circ) = -\sin(90^\circ - A) = -\cos A.$
19. $\sec(A - 180^\circ) = \sec(180^\circ - A) = -\sec A.$
20. $\sin(270^\circ - A) = \sin(180^\circ + 90^\circ - A) = -\sin(90^\circ - A) = -\cos A.$
21. $\cot(A - 90^\circ) = -\cot(90^\circ - A) = -\tan A.$
22. $\sin \left(\theta - \frac{\pi}{2}\right) = -\sin \left(\frac{\pi}{2} - \theta\right) = -\cos \theta.$
23. $\tan(\theta - \pi) = -\tan(\pi - \theta) = \tan \theta.$
24. $\sec \left(\frac{3\pi}{2} - \theta\right) = \sec \left(\pi + \frac{\pi}{2} - \theta\right) = -\sec \left(\frac{\pi}{2} - \theta\right) = -\operatorname{cosec} \theta.$
25. Expression = $\tan A \cos A \operatorname{cosec} A = 1.$
26. Expression = $-\sin A + \sin A - (-\sin A) - (-\sin A) = 2 \sin A.$
27. Expression = $\sec^2 A - \tan^2 A = 1.$

EXAMPLES. X. b. PAGE 91.

1. $\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ = -\frac{1}{2}.$

2. $\sin 960^\circ = \sin(3 \times 360^\circ - 120^\circ) = -\sin 120^\circ = -\frac{\sqrt{3}}{2}.$

3. $\cos 780^\circ = \cos(2 \times 360^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}.$

4. $\sin(-870^\circ) = \sin(-2 \times 360^\circ - 150^\circ) = -\sin 150^\circ = -\frac{1}{2}.$
5. $\sec 900^\circ = \sec(2 \times 360^\circ + 180^\circ) = \sec 180^\circ = -1.$
6. $\tan(-855^\circ) = -\tan 855^\circ = -\tan(2 \times 360^\circ + 135^\circ)$
 $= -\tan 135^\circ = -\tan(180^\circ - 45^\circ) = \tan 45^\circ = 1.$
7. $\operatorname{cosec}(-660^\circ) = -\operatorname{cosec} 660^\circ = -\operatorname{cosec}(2 \times 360^\circ - 60^\circ) = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$
8. $\cot 840^\circ = \cot(2 \times 360^\circ + 120^\circ) = \cot 120^\circ$
 $= \cot(180^\circ - 60^\circ) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}.$
9. $\operatorname{cosec}(-765^\circ) = -\operatorname{cosec} 765^\circ$
 $= -\operatorname{cosec}(2 \times 360^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$
10. $\cos 1125^\circ = \cos(3 \times 360^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}.$
11. $\cot 990^\circ = \cot(3 \times 360^\circ - 90^\circ) = -\cot 90^\circ = 0.$
12. $\sin 855^\circ = \sin(2 \times 360^\circ + 135^\circ) = \sin 135^\circ = \sin(180^\circ - 45^\circ)$
 $= \sin 45^\circ = \frac{1}{\sqrt{2}}.$
13. $\sec 1305^\circ = \sec(4 \times 360^\circ - 135^\circ) = \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
14. $\cos 960^\circ = \cos(3 \times 360^\circ - 120^\circ) = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}.$
15. $\sec(-1575^\circ) = \sec 1575^\circ = \sec(4 \times 360^\circ + 135^\circ)$
 $= \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
16. $\sin \frac{15\pi}{4} = \sin\left(4\pi - \frac{\pi}{4}\right) = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
17. $\cot \frac{23\pi}{4} = \cot\left(6\pi - \frac{\pi}{4}\right) = -\cot \frac{\pi}{4} = -1.$
18. $\sec \frac{7\pi}{3} = \sec\left(2\pi + \frac{\pi}{3}\right) = \sec \frac{\pi}{3} = 2.$
19. $\cot \frac{16\pi}{3} = \cot\left(6\pi - \frac{2\pi}{3}\right) = -\cot \frac{2\pi}{3} = \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}.$
20. $\sec\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) = \sec\left(2\pi + \frac{\pi}{3} - \frac{\pi}{2}\right) = \sec\left(2\pi - \frac{\pi}{6}\right) = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}.$
21. $\cos \theta = \frac{\sqrt{3}}{2} = \cos 30^\circ; \therefore \theta = 30^\circ \text{ satisfies the equation.}$

And $\cos 30^\circ = \cos(360^\circ - 30^\circ) = \cos 330^\circ.$

There are no angles whose boundary lines are in the second and third

quadrants which satisfy the equation since the cosine in those quadrants is negative.

\therefore the positive angles are $30^\circ, 330^\circ$.

And the negative angles are $-(360^\circ - 30^\circ), -(360^\circ - 330^\circ)$.

That is, the angles are $\pm 30^\circ, \pm 330^\circ$.

$$22. \sin \theta = -\frac{1}{2} = \sin (180^\circ + 30^\circ) = \sin 210^\circ; \quad \therefore \theta = 210^\circ \text{ is a solution.}$$

$$\text{Also } \sin (360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}; \quad \therefore \theta = 330^\circ \text{ is another solution.}$$

Thus $210^\circ, 330^\circ$ are the positive angles.

The negative angles are $-(360^\circ - 210^\circ), -(360^\circ - 330^\circ)$;

\therefore the required angles are $210^\circ, 330^\circ, -150^\circ, -30^\circ$.

$$23. \tan \theta = -\sqrt{3} = \tan (180^\circ - 60^\circ) = \tan 120^\circ; \quad \therefore \theta = 120^\circ \text{ is a solution.}$$

$$\text{Also } \tan (360^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}; \quad \therefore \theta = 300^\circ \text{ is another solution.}$$

Thus $120^\circ, 300^\circ$ are the positive angles.

The negative angles are $-(360^\circ - 120^\circ), -(360^\circ - 300^\circ)$;

\therefore the required angles are $120^\circ, 300^\circ, -240^\circ, -60^\circ$.

$$24. \cot \theta = -1 = -\cot 45^\circ = \cot 135^\circ.$$

$$\text{Also } \cot 135^\circ = \cot (180^\circ + 135^\circ) = \cot 315^\circ.$$

\therefore the positive angles which satisfy the equation are $135^\circ, 315^\circ$.

The negative angles are $-(360^\circ - 135^\circ), -(360^\circ - 315^\circ)$.

\therefore the required angles are $135^\circ, 315^\circ, -45^\circ, -225^\circ$.

25. Let the radius vector OP start from the position OX and revolve in the positive direction till it reaches the position OP , such that $\angle POX = A$. Then let it revolve in the negative direction through an angle of 180° , reaching the position OP' .

Then POP' is a straight line, and $\angle XOP' = A - 180^\circ$.

Draw $PM, P'M'$ perpendicular to XX' . Then the $\Delta^* OPM, OP'M'$ are geometrically equal.

$$\text{Then } \sec(A - 180^\circ) = \frac{OP'}{OM'} = -\frac{OP}{OM} = -\sec A.$$

26. Proceed as in Art. 97. Let the radius vector first revolve from OX through the angle A to the position OP . Again, let it revolve from OX through 270° and then further through an angle A to the position OP' ; draw $PM, P'M'$ perpendiculars to XX' . Then from the equal $\Delta^* OPM, OP'M'$, we have

$$P'M' = -OM, \quad O'M' = PM;$$

$$\therefore \tan(270^\circ + A) = \frac{P'M'}{O'M'} = -\frac{OM}{PM} = -\cot A.$$

27. Let OP be determined as before, and then let the radius vector turn back in the negative direction through an angle 90° to the position OP' . Draw perpendiculars as before.

$$\text{Then } \cos(A - 90^\circ) = \frac{OM'}{OP'} = \frac{PM}{OP} = \sin A.$$

$$28. \text{ First side} = \tan A - \tan A - \tan A = -\tan A = \tan(360^\circ - A).$$

$$29. \text{ First side} = \frac{\sin A}{\tan A} \cdot \frac{\tan A}{-\cot A} \cdot \frac{\cos A}{-\sin A} = \sin A.$$

$$30. \text{ Expression} = \frac{-\sin A}{-\sin A} - \frac{-\cot A}{\cot A} + \frac{\cos A}{\cos A} = 1 + 1 + 1 = 3.$$

$$31. \text{ Expression} = \frac{\operatorname{cosec} A}{-\sec A} \cdot \frac{\cos A}{-\sin A} = \frac{\cos^2 A}{\sin^2 A} = \cot^2 A.$$

$$32. \text{ Expression} = \frac{-\sin A \cdot \sec A \cdot (-\tan A)}{\sec A (-\sin A) \tan A} = -1.$$

$$33. \text{ First side} = \sin\left(\pi - \frac{\pi}{2} - \theta\right) \sin\left(\frac{\pi}{2} + \pi - \theta\right) \cot\left(\pi + \frac{\pi}{2} + \theta\right) \\ = \sin\left(\frac{\pi}{2} - \theta\right) \sin\left(\frac{3\pi}{2} - \theta\right) \cot\left(\frac{\pi}{2} + \theta\right).$$

$$34. \sin \alpha = \sin \frac{11\pi}{4} = \sin\left(2\pi + \frac{3\pi}{4}\right) = \sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}};$$

$$\cos \alpha = \cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}. \quad \text{Also } \tan \alpha = -1.$$

$$\therefore \text{Expression} = \frac{1}{2} - \frac{1}{2} - 2 - 2 = -4.$$

EXAMPLES. XI. a. PAGE 97.

$$1. \sin(A + 45^\circ) = \sin A \cos 45^\circ + \cos A \sin 45^\circ = \frac{1}{\sqrt{2}} (\sin A + \cos A).$$

$$2. \cos(A + 45^\circ) = \cos A \cos 45^\circ - \sin A \sin 45^\circ = \frac{1}{\sqrt{2}} (\cos A - \sin A).$$

$$3. 2 \sin(30^\circ - A) = 2 (\sin 30^\circ \cos A - \cos 30^\circ \sin A) = \cos A - \sqrt{3} \sin A.$$

$$4. \cos A = \frac{4}{5}; \quad \therefore \sin A = \frac{3}{5}. \quad \cos B = \frac{3}{5}; \quad \therefore \sin B = \frac{4}{5}.$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B = 1;$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{24}{25}.$$

$$6. \sec A = \frac{17}{8}; \quad \therefore \cos A = \frac{8}{17}, \quad \sin A = \frac{15}{17}.$$

$$\operatorname{cosec} B = \frac{5}{4}; \quad \therefore \sin B = \frac{4}{5}, \quad \cos B = \frac{3}{5}.$$

$$\therefore \sec(A+B) = \frac{1}{\cos A \cos B - \sin A \sin B} = -\frac{85}{36}.$$

$$7. \sin 75^\circ = \sin(90^\circ - 15^\circ) = \cos 15^\circ = \cos(45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$8. \sin 15^\circ = \cos(90^\circ - 15^\circ) = \cos 75^\circ = \cos(45^\circ + 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$9. \frac{\sin(\alpha+\beta)}{\cos \alpha \cos \beta} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta.$$

$$10. \frac{\sin(\alpha-\beta)}{\sin \alpha \sin \beta} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \sin \beta} = \cot \beta - \cot \alpha.$$

$$11. \frac{\cos(\alpha-\beta)}{\cos \alpha \sin \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \sin \beta} = \cot \beta + \tan \alpha.$$

$$12. \cos(A+B) \cos(A-B) \\ = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ = \cos^2 A - \sin^2 B.$$

$$13. \sin(A+B) \sin(A-B)$$

$$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ = (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) \\ = \cos^2 B - \cos^2 A.$$

$$14. \cos(45^\circ - A) - \sin(45^\circ + A) = \frac{1}{\sqrt{2}} \{ \cos A + \sin A - \sin A - \cos A \} = 0.$$

$$15. \cos(45^\circ + A) + \sin(A - 45^\circ) = \frac{1}{\sqrt{2}} (\cos A - \sin A + \sin A - \cos A) = 0.$$

$$16. \text{First side} = \cos A \cos B + \sin A \sin B - \sin A \cos B - \cos A \sin B \\ = (\cos A - \sin A) \cos B - (\cos A - \sin A) \sin B \\ = (\cos A - \sin A)(\cos B - \sin B).$$

17. First side = $\cos A \cos B - \sin A \sin B + \sin A \cos B - \cos A \sin B$
 $= (\cos A + \sin A) \cos B - (\cos A + \sin A) \sin B$
 $= (\cos A + \sin A) (\cos B - \sin B).$

18. First side

$$\begin{aligned} &= 2(\sin A \cos 45^\circ + \cos A \sin 45^\circ)(\sin A \cos 45^\circ - \cos A \sin 45^\circ) \\ &= 2 \times \frac{1}{\sqrt{2}}(\sin A + \cos A) \times \frac{1}{\sqrt{2}}(\sin A - \cos A) \\ &= \sin^2 A - \cos^2 A. \end{aligned}$$

19. First side = $2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha - \sin \alpha) \times \frac{1}{\sqrt{2}}(\cos \alpha + \sin \alpha)$
 $= \cos^2 \alpha - \sin^2 \alpha.$

20. First side = $2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha + \sin \alpha) \times \frac{1}{\sqrt{2}}(\cos \beta - \sin \beta)$
 $= \cos \alpha \cos \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta - \sin \alpha \sin \beta$
 $= \cos(\alpha + \beta) + \sin(\alpha - \beta).$

21. As in Ex. 10, it is easily shewn that the first term of the expression
 $= \tan \beta - \tan \gamma.$

Thus the first side

$$= \tan \beta - \tan \gamma + \tan \gamma - \tan \alpha + \tan \alpha - \tan \beta = 0.$$

EXAMPLES. XI. b. PAGE 100.

1. We have $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$;

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.$$

3. We have $\cot A = \frac{5}{7}$, $\cot B = \frac{7}{5}$;

$$\therefore \cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} = 0,$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{7}{5} - \frac{5}{7}}{1 + \frac{1}{5}} = \frac{12}{35}.$$

5. $\tan(45^\circ + A) = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \frac{1 + \tan A}{1 - \tan A}.$

$$6. \tan(45^\circ - A) = \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} = \frac{1 - \tan A}{1 + \tan A}.$$

$$7. \cot\left(\frac{\pi}{4} - \theta\right) = \frac{\cot\frac{\pi}{4} \cot \theta + 1}{\cot \theta - \cot\frac{\pi}{4}} = \frac{\cot \theta + 1}{\cot \theta - 1}.$$

$$8. \cot\left(\frac{\pi}{4} + \theta\right) = \frac{\cot\frac{\pi}{4} \cot \theta - 1}{\cot \theta + \cot\frac{\pi}{4}} = \frac{\cot \theta - 1}{\cot \theta + 1}.$$

$$9. \tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}.$$

$$10. \cot 15^\circ = \cot(45^\circ - 30^\circ) = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}.$$

$$11. \cos(A + B + C) = \cos A \cos(B + C) - \sin A \sin(B + C) \\ = \cos A \cos B \cos C - \cos A \sin B \sin C \\ - \sin A \sin B \cos C - \sin A \cos B \sin C.$$

$$\sin(A + B + C) = \sin(A - B) \cos C + \cos(A - B) \sin C \\ = \sin A \cos B \cos C - \cos A \sin B \cos C \\ + \cos A \cos B \sin C + \sin A \sin B \sin C.$$

$$12. \tan(A - B - C) = \frac{\tan(A - B) - \tan C}{1 + \tan(A - B) \tan C} \\ = \frac{\tan A - \tan B}{1 + \tan A \tan B} - \tan C \\ = \frac{(\tan A - \tan B) \tan C}{1 + \tan A \tan B} \\ = \frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B - \tan B \tan C + \tan C \tan A}.$$

$$13. \cot(A + B + C) = \frac{\cot(A + B) \cot C - 1}{\cot C + \cot(A + B)} \\ = \frac{(\cot A \cot B - 1) \cot C}{\cot B + \cot A} - 1 \\ = \frac{\cot A \cot B - 1}{\cot C + \frac{\cot A \cot B - 1}{\cot B + \cot A}} \\ = \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}.$$

EXAMPLES. XI. c. PAGE 101.

1. First side = $\cos(\overline{A+B} - \overline{B}) = \cos A.$
2. First side = $\sin(3\overline{A} - \overline{A}) = \sin 2A.$
3. First side = $\cos(2\alpha - \alpha) = \cos \alpha.$
4. First side = $\cos(30^\circ + \overline{A} + 30^\circ - \overline{A}) = \cos 60^\circ = \frac{1}{2}.$
5. First side = $\sin(60^\circ - \overline{A} + 30^\circ + \overline{A}) = \sin 90^\circ = 1.$
6. First side = $\cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha = \cos(2\alpha + \alpha) = \cos 3\alpha.$
7. First side = $\tan(\overline{\alpha - \beta} + \overline{\beta}) = \tan \alpha.$
8. First side = $\cot(\overline{\alpha + \beta} - \overline{\alpha}) = \cot \beta.$
9. First side = $\tan(4\overline{A} - 3\overline{A}) = \tan A.$
10. $\cot \theta - \cot 2\theta = \frac{\cos \theta}{\sin \theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta \cos \theta - \cos 2\theta \sin \theta}{\sin \theta \sin 2\theta}$
 $= \frac{\sin(2\theta - \theta)}{\sin \theta \sin 2\theta} = \frac{\sin \theta}{\sin \theta \sin 2\theta} = \text{cosec } 2\theta.$
11. $1 + \tan 2\theta \tan \theta = 1 + \frac{\sin 2\theta \sin \theta}{\cos 2\theta \cos \theta} = \frac{\cos \theta \sin 2\theta + \sin \theta \sin 2\theta}{\cos \theta \cos 2\theta}$
 $= \frac{\cos(2\theta - \theta)}{\cos \theta \cos 2\theta} = \sec 2\theta.$
12. $1 + \cot 2\theta \cot \theta = \frac{\sin 2\theta \sin \theta + \cos 2\theta \cos \theta}{\sin \theta \sin 2\theta}$
 $= \frac{\cos(2\theta - \theta)}{\sin \theta \sin 2\theta} = \text{cosec } 2\theta \cot \theta.$
13. First side = $\sin(2\theta + \theta) = \sin 3\theta$
 $= \sin(4\theta - \theta)$
 $= \sin 4\theta \cos \theta - \cos 4\theta \sin \theta.$
14. First side = $\cos(4\alpha + \alpha) = \cos 5\alpha$
 $= \cos(3\alpha + 2\alpha)$
 $= \cos 3\alpha \cos 2\alpha - \sin 3\alpha \sin 2\alpha.$

EXAMPLES. XI. d. PAGE 104.

1. Here $\cos 2A = 2\cos^2 A - 1 = -\frac{7}{9}.$

3. We have $\sin A = \frac{3}{5}$, and $\cos A = \frac{4}{5}$;

$$\therefore \sin 2A = 2 \sin A \cos A = \frac{24}{25}.$$

5. By Art. 124, $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{7}{25}$,

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{24}{25}.$$

7. See Example, Art. 122.

$$8. \frac{\sin 2A}{1 + \cos 2A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \tan A.$$

$$9. \frac{\sin 2A}{1 - \cos 2A} = \frac{2 \sin A \cos A}{2 \sin^2 A} = \cot A.$$

$$10. \frac{1 - \cos A}{\sin A} = \frac{2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \tan \frac{A}{2}.$$

$$11. \frac{1 + \cos A}{\sin A} = \frac{2 \cos^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \cot \frac{A}{2}.$$

$$12. 2 \operatorname{cosec} 2a = \frac{2}{\sin 2a} = \frac{1}{\sin a \cos a} = \sec a \operatorname{cosec} a.$$

$$13. \tan a + \cot a = \frac{\sin^2 a + \cos^2 a}{\sin a \cos a} = \frac{1}{\sin a \cos a} = 2 \operatorname{cosec} 2a.$$

$$14. \cos^4 a - \sin^4 a = (\cos^2 a + \sin^2 a)(\cos^2 a - \sin^2 a) = \cos 2a.$$

$$15. \cot a - \tan a = \frac{\cos^2 a - \sin^2 a}{\sin a \cos a} = \frac{2 \cos 2a}{2 \sin a \cos a} = 2 \cot 2a.$$

$$16. \text{By Art. 116, } \cot 2A = \frac{\cot A \cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2 \cot A}.$$

$$17. \frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{1}{\tan A} - \tan A}{\frac{1}{\tan A} + \tan A} = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A.$$

$$18. \frac{1 + \cot^2 A}{2 \cot A} = \frac{\sin^2 A + \cos^2 A}{2 \cot A \sin^2 A} = \frac{1}{2 \sin A \cos A} = \operatorname{cosec} 2A.$$

$$19. \frac{\cot^2 A + 1}{\cot^2 A - 1} = \frac{1 + \tan^2 A}{1 - \tan^2 A} = \sec 2A.$$

$$20. \frac{1 + \sec \theta}{\sec \theta} = \frac{1}{\sec \theta} + 1 = \cos \theta + 1 = 2 \cos^2 \frac{\theta}{2}.$$

$$21. \frac{\sec \theta - 1}{\sec \theta} = 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}.$$

$$22. \frac{2 - \sec^2 \theta}{\sec^2 \theta} = \frac{2}{\sec^2 \theta} - 1 = 2 \cos^2 \theta - 1 = \cos 2\theta.$$

$$23. \frac{\operatorname{cosec}^2 \theta - 2}{\operatorname{cosec}^2 \theta} = 1 - 2 \sin^2 \theta = \cos 2\theta.$$

$$24. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 + \sin A.$$

$$25. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 - \sin A.$$

$$26. \frac{\cos 2a}{1 + \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\sin a + \cos a)^2} = \frac{\cos a - \sin a}{\cos a + \sin a} = \frac{1 - \tan a}{1 + \tan a} = \tan(45^\circ - a).$$

$$27. \frac{\cos 2a}{1 - \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\cos a - \sin a)^2} = \frac{\cos a + \sin a}{\cos a - \sin a} = \cot(45^\circ - a).$$

$$28. \sin 8A = 2 \sin 4A \cos 4A = 4 \sin 2A \cos 2A \cos 4A \\ = 8 \sin A \cos A \cos 2A \cos 4A.$$

$$29. \cos 4A = 2 \cos^2 2A - 1 \\ = 2(2 \cos^2 A - 1)^2 - 1 \\ = 8 \cos^4 A - 8 \cos^2 A + 1.$$

$$30. \text{Second side} = \cos 2 \left(45^\circ - \frac{A}{2}\right) = \cos(90^\circ - A) = \sin A.$$

$$31. \cos^2 \left(\frac{\pi}{4} - a\right) - \sin^2 \left(\frac{\pi}{4} - a\right) = \cos 2 \left(\frac{\pi}{4} - a\right) = \cos \left(\frac{\pi}{2} - 2a\right) = \sin 2a.$$

$$32. \text{First side} = \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{4 \tan A}{1 - \tan^2 A} = 2 \tan 2A.$$

$$33. \text{First side} = \frac{1 + \tan A}{1 - \tan A} + \frac{1 - \tan A}{1 + \tan A} = \frac{2(1 + \tan^2 A)}{1 - \tan^2 A} \\ = 2 \sec 2A. \quad [\text{Art. 124.}]$$

EXAMPLES. XI. e. PAGE 106.

$$4. \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3 \sin A - 4 \sin^3 A}{\sin A} - \frac{4 \cos^3 A - 3 \cos A}{\cos A}$$

$$= 3 - 4 \sin^2 A - 4 \cos^2 A + 3$$

$$= 6 - 4 (\sin^2 A + \cos^2 A) = 2.$$

$$5. \cot 3A = \cot (2A + A) = \frac{\cot 2A \cdot \cot A - 1}{\cot 2A + \cot A}$$

$$= \frac{\cot^2 A - 1}{\frac{2 \cot A}{\cot^2 A - 1} \cdot \cot A - 1} = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}.$$

$$6. \text{First side} = \frac{3 \cos \alpha + 4 \cos^3 \alpha - 3 \cos \alpha}{3 \sin \alpha - 3 \sin \alpha + 4 \sin^3 \alpha} = \cot^3 \alpha.$$

$$7. \text{First side} = \frac{3 \sin \alpha - 3 \sin^3 \alpha}{3 \cos \alpha - 3 \cos^3 \alpha} = \frac{\sin \alpha (1 - \sin^2 \alpha)}{\cos \alpha (1 - \cos^2 \alpha)} = \cot \alpha.$$

$$8. \text{First side} = \frac{3 \cos \alpha - 3 \cos^3 \alpha}{\cos \alpha} + \frac{3 \sin \alpha - 3 \sin^3 \alpha}{\sin \alpha}$$

$$= 3 - 3 \cos^2 \alpha + 3 - 3 \sin^2 \alpha$$

$$= 6 - 3 (\cos^2 \alpha + \sin^2 \alpha) = 3.$$

$$9. \sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

$$10. \cos 36^\circ - \sin 18^\circ = \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} = \frac{1}{2}.$$

$$11. \cos^2 36^\circ + \sin^2 18^\circ = \left(\frac{\sqrt{5}+1}{4}\right)^2 + \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{3+\sqrt{5}}{8} + \frac{3-\sqrt{5}}{8} = \frac{3}{4}.$$

$$12. 4 \sin 18^\circ \cos 36^\circ = 4 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} = \frac{5-1}{4} = 1.$$

EXAMPLES. XI. f. PAGE 108.

$$1. \text{First side} = \frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} = \frac{\sin 2A \cos A - \cos 2A \sin A}{\cos A \cos 2A}$$

$$= \frac{\sin (2A - A)}{\cos A \cos 2A} = \tan A \sec 2A.$$

$$2. \text{First side} = \frac{\sin 2A}{\cos 2A} + \frac{\cos A}{\sin A} = \frac{\sin 2A \sin A + \cos 2A \cos A}{\cos 2A \sin A}$$

$$= \frac{\cos A}{\cos 2A \sin A} = \cot A \sec 2A.$$

$$3. \text{ First side} = \frac{2 \sin^2 \theta + 2 \sin \theta \cos \theta}{2 \cos^2 \theta + 2 \sin \theta \cos \theta} = \frac{2 \sin \theta (\sin \theta + \cos \theta)}{2 \cos \theta (\sin \theta + \cos \theta)} = \tan \theta.$$

$$4. \text{ First side} = \frac{2 \cos^2 \frac{\theta}{2} + \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}.$$

$$\begin{aligned} 5. \cos^6 \alpha - \sin^6 \alpha &= (\cos^2 \alpha - \sin^2 \alpha)(\cos^4 \alpha + \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha) \\ &= \cos 2\alpha [(\cos^2 \alpha + \sin^2 \alpha)^2 - \sin^2 \alpha \cos^2 \alpha] \\ &= \cos 2\alpha \left(1 - \frac{1}{4} \sin^2 2\alpha\right). \end{aligned}$$

$$\begin{aligned} 6. 4(\cos^6 \theta + \sin^6 \theta) &= 4(\cos^2 \theta + \sin^2 \theta)(\cos^4 \theta + \sin^4 \theta - \sin^2 \theta \cos^2 \theta) \\ &= 4\{(\cos^2 \theta + \sin^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta\} \\ &= 4 - 3 \sin^2 2\theta = 1 + 3(1 - \sin^2 2\theta) \\ &= 1 + 3 \cos^2 2\theta. \end{aligned}$$

$$\begin{aligned} 7. \text{ First side} &= \frac{4 \cos^3 \alpha - 3 \cos \alpha + 3 \sin \alpha - 4 \sin^3 \alpha}{\cos \alpha - \sin \alpha} \\ &= \frac{4(\cos^3 \alpha - \sin^3 \alpha) - 3(\cos \alpha - \sin \alpha)}{\cos \alpha - \sin \alpha} \\ &= 4(\cos^2 \alpha + \sin \alpha \cos \alpha + \sin^2 \alpha) - 3 \\ &= 4 + 2 \sin 2\alpha - 3 = 1 + 2 \sin 2\alpha. \end{aligned}$$

$$\begin{aligned} 8. \text{ First side} &= \frac{4 \cos^3 \alpha - 3 \cos \alpha - 3 \sin \alpha + 4 \sin^3 \alpha}{\cos \alpha + \sin \alpha} \\ &= 4(\cos^2 \alpha - \cos \alpha \sin \alpha + \sin^2 \alpha) - 3 \\ &= 4 - 2 \sin 2\alpha - 3 = 1 - 2 \sin 2\alpha. \end{aligned}$$

$$9. \frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} = \frac{(\cos \alpha + \sin \alpha)^2}{\cos^2 \alpha - \sin^2 \alpha} = \frac{\sin 2\alpha + 1}{\cos 2\alpha} = \tan 2\alpha + \sec 2\alpha.$$

$$10. \frac{\cot \alpha - 1}{\cot \alpha + 1} = \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} = \frac{1 - \sin 2\alpha}{\cos 2\alpha}. \quad [\text{See Example 2, p. 107.}]$$

$$11. \frac{1 + \sin \theta}{\cos \theta} = \frac{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)^2}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}.$$

$$12. \frac{\cos \theta}{1 - \sin \theta} = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{\cot \frac{\theta}{2} + 1}{\cot \frac{\theta}{2} - 1}.$$

$$13. \sec A - \tan A = \frac{1 - \sin A}{\cos A} = \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}} = \tan \left(45^\circ - \frac{A}{2} \right).$$

$$14. \tan A + \sec A = \frac{\sin A + 1}{\cos A} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}} = \cot \left(45^\circ - \frac{A}{2} \right).$$

$$15. \text{Second side} = \frac{2 \sin^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right)}{2 \cos^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right)} = \frac{1 - \cos \left(\frac{\pi}{2} + \theta \right)}{1 + \cos \left(\frac{\pi}{2} + \theta \right)} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$

$$16. (2 \cos A + 1)(2 \cos A - 1) = 4 \cos^2 A - 1 = 2 \cos 2A + 1.$$

$$17. \text{First side} = \frac{2 \sin A \cos A}{2 \cos^2 A} \times \frac{\cos A}{2 \cos^2 \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \tan \frac{A}{2}.$$

$$18. \text{First side} = \frac{2 \sin A \cos A}{2 \sin^2 A} \times \frac{2 \sin^2 \frac{A}{2}}{\cos A} = \frac{2 \sin^2 \frac{A}{2}}{\sin A} = \tan \frac{A}{2}.$$

$$19. \text{First side} = \cos 3a (3 \sin a - \sin 3a) + \sin 3a (3 \cos a + \cos 3a) \\ = 3 (\cos 3a \sin a + \sin 3a \cos a) = 3 \sin 4a.$$

$$20. \text{First side} = \frac{1}{4} (3 \cos a + \cos 3a) \cos 3a + \frac{1}{4} (3 \sin a - \sin 3a) \sin 3a \\ = \frac{3}{4} (\cos a \cos 3a + \sin a \sin 3a) + \frac{1}{4} (\cos^2 3a - \sin^2 3a) \\ = \frac{1}{4} \{3 \cos(3a - a) + \cos 6a\} = \frac{1}{4} (3 \cos 2a + \cos 6a) = \cos^3 2a.$$

$$21. \text{First side} = 3 \cos 20^\circ + \cos 60^\circ + 3 \cos 40^\circ + \cos 120^\circ \\ = 3 (\cos 20^\circ + \cos 40^\circ).$$

$$22. \text{First side} = 3 \cos 10^\circ + \cos 30^\circ + 3 \sin 20^\circ - \sin 60^\circ \\ = 3 (\cos 10^\circ + \sin 20^\circ).$$

$$23. \tan 3A = \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}.$$

Multiply up and transpose; then we obtain

$$\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$$

$$\begin{aligned}
 24. \quad & \frac{\cot \theta}{\cot \theta - \cot 3\theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} = \frac{\frac{1}{\tan \theta}}{\frac{1}{\tan \theta} - \frac{1}{\tan 3\theta}} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} \\
 &= \frac{\tan 3\theta}{\tan 3\theta - \tan \theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} = \frac{\tan 3\theta - \tan \theta}{\tan 3\theta - \tan \theta} = 1.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \frac{1}{\tan 3\theta + \tan \theta} - \frac{1}{\cot 3\theta + \cot \theta} = \frac{1}{\tan 3\theta + \tan \theta} - \frac{\tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} \\
 &= \frac{1 - \tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} = \frac{1}{\tan 4\theta} = \cot 4\theta.
 \end{aligned}$$

EXAMPLES. XII. a. PAGE 112.

For Examples 1—16, see Examples on page 111.

$$\begin{aligned}
 17. \quad 2 \cos 2\beta \cos (\alpha - \beta) &= \cos (2\beta + \alpha - \beta) + \cos (2\beta - \alpha + \beta) \\
 &= \cos (\beta + \alpha) + \cos (3\beta - \alpha).
 \end{aligned}$$

$$\begin{aligned}
 18. \quad 2 \sin 3\alpha \sin (\alpha + \beta) &= \cos (3\alpha - \alpha - \beta) - \cos (3\alpha + \alpha + \beta) \\
 &= \cos (2\alpha - \beta) - \cos (4\alpha + \beta).
 \end{aligned}$$

$$\begin{aligned}
 19. \quad 2 \sin (2\theta + \phi) \cos (\theta - 2\phi) &= \sin (2\theta + \phi + \theta - 2\phi) + \sin (2\theta + \phi - \theta + 2\phi) \\
 &= \sin (3\theta - \phi) + \sin (\theta + 3\phi).
 \end{aligned}$$

$$\begin{aligned}
 20. \quad 2 \cos (3\theta + \phi) \sin (\theta - 2\phi) &= \sin (3\theta + \phi + \theta - 2\phi) - \sin (3\theta + \phi - \theta + 2\phi) \\
 &= \sin (4\theta - \phi) - \sin (2\theta + 3\phi).
 \end{aligned}$$

$$21. \quad \cos (60^\circ + \alpha) \sin (60^\circ - \alpha) = \frac{1}{2} \{ \sin 120^\circ - \sin 2\alpha \} = \frac{1}{2} \left(\frac{\sqrt{3}}{2} - \sin 2\alpha \right).$$

EXAMPLES. XII. b. PAGE 114.

For Examples 1—12, see Examples on page 113.

$$13. \quad \frac{\cos \alpha - \cos 3\alpha}{\sin 3\alpha - \sin \alpha} = \frac{2 \sin 2\alpha \sin \alpha}{2 \cos 2\alpha \sin \alpha} = \tan 2\alpha.$$

$$14. \quad \frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha} = \frac{2 \sin \frac{5\alpha}{2} \cos \frac{\alpha}{2}}{2 \sin \frac{5\alpha}{2} \sin \frac{\alpha}{2}} = \cot \frac{\alpha}{2}.$$

$$15. \quad \frac{\cos 4\theta - \cos \theta}{\sin \theta - \sin 4\theta} = \frac{-2 \sin \frac{5\theta}{2} \sin \frac{3\theta}{2}}{-2 \cos \frac{5\theta}{2} \sin \frac{3\theta}{2}} = \tan \frac{5\theta}{2}.$$

16. $\frac{\cos 2\theta - \cos 12\theta}{\sin 12\theta + \sin 2\theta} = \frac{2 \sin 7\theta \sin 5\theta}{2 \sin 7\theta \cos 5\theta} = \tan 5\theta.$

17. First side $= 2 \cos 60^\circ \sin A = 2 \times \frac{1}{2} \sin A = \sin A.$

18. First side $= 2 \cos 30^\circ \cos A = \sqrt{3} \cos A.$

19. First side $= 2 \sin \frac{\pi}{4} \sin (-a) = -2 \times \frac{1}{\sqrt{2}} \sin a = -\sqrt{2} \sin a.$

20. First side $= \frac{2 \cos \alpha \cos (\alpha - 3\beta)}{2 \sin \alpha \cos (\alpha - 3\beta)} = \cot \alpha.$

21. First side $= \frac{2 \sin (2\theta - \phi) \sin (\theta + 2\phi)}{2 \sin (2\theta - \phi) \cos (\theta + 2\phi)} = \tan (\theta + 2\phi).$

22. First side $= \frac{2 \cos \frac{\alpha + 5\beta}{2} \sin \frac{\alpha - 3\beta}{2}}{2 \cos \frac{\alpha + 5\beta}{2} \cos \frac{\alpha - 3\beta}{2}} = \tan \frac{\alpha - 3\beta}{2}.$

EXAMPLES. XII. c. PAGE 116.

1. $\cos 3A + \sin 2A - \sin 4A = \cos 3A - 2 \cos 3A \sin A$
 $= \cos 3A (1 - 2 \sin A).$

2. $\sin 3\theta - \sin \theta - \sin 5\theta = \sin 3\theta - 2 \sin 3\theta \cos 2\theta$
 $= \sin 3\theta (1 - 2 \cos 2\theta).$

3. $\cos \theta + \cos 2\theta + \cos 5\theta = \cos 2\theta + 2 \cos 3\theta \cos 2\theta$
 $= \cos 2\theta (1 + 2 \cos 3\theta).$

4. $\sin a - \sin 2a + \sin 3a = \sin a + 2 \cos \frac{5a}{2} \sin \frac{a}{2}$
 $= 2 \sin \frac{a}{2} \left(\cos \frac{a}{2} + \cos \frac{5a}{2} \right)$
 $= 4 \sin \frac{a}{2} \cos a \cos \frac{3a}{2}.$

5. $\sin 3a + \sin 7a + \sin 10a = 2 \sin 5a \cos 2a + \sin 10a$
 $= 2 \sin 5a (\cos 2a + \cos 5a)$
 $= 2 \sin 5a \cos \frac{7a}{2} \cos \frac{3a}{2}.$

6. $\sin A + 2 \sin 3A + \sin 5A = 2 \sin 3A + 2 \sin 3A \cos 2A$
 $= 2 \sin 3A (1 + \cos 2A)$
 $= 4 \sin 3A \cos^2 A.$

7. First side = $\frac{\sin 2\alpha + 2 \cos 3\alpha \sin 2\alpha}{\cos 2\alpha + 2 \cos 3\alpha \cos 2\alpha} = \frac{\sin 2\alpha (1 + 2 \cos 3\alpha)}{\cos 2\alpha (1 + 2 \cos 3\alpha)} = \tan 2\alpha.$

8. First side = $\frac{(\sin \alpha + \sin 5\alpha) + (\sin 2\alpha + \sin 4\alpha)}{(\cos \alpha + \cos 5\alpha) + (\cos 2\alpha + \cos 4\alpha)}$
 $= \frac{2 \sin 3\alpha \cos 2\alpha + 2 \sin 3\alpha \cos \alpha}{2 \cos 3\alpha \cos 2\alpha + 2 \cos 3\alpha \cos \alpha} = \frac{\sin 3\alpha}{\cos 3\alpha} = \tan 3\alpha.$

9. First side = $\frac{2 \cos 5\theta \cos 2\theta - 2 \cos 3\theta \cos 2\theta}{2 \cos 5\theta \sin 2\theta - 2 \cos 3\theta \sin 2\theta} = \cot 2\theta.$

10. First side = $\frac{1}{2} (\sin 5A - \sin A) - \frac{1}{2} (\sin 5A - \sin 3A)$
 $= \frac{1}{2} (\sin 3A - \sin A) = \cos 2A \sin A.$

11. First side = $\frac{1}{2} (\cos 7A + \cos 3A) - \frac{1}{2} (\cos 7A + \cos A)$
 $= \frac{1}{2} (\cos 3A - \cos A) = -\sin 2A \sin A.$

12. First side = $\frac{1}{2} (\sin 5\theta + \sin 3\theta) - \frac{1}{2} (\sin 5\theta + \sin \theta)$
 $= \frac{1}{2} (\sin 3\theta - \sin \theta) = \cos 2\theta \sin \theta.$

13. $\cos 5^\circ - \sin 25^\circ = \cos 5^\circ - \cos 65^\circ$
 $= 2 \sin 35^\circ \sin 30^\circ = \sin 35^\circ.$

14. $\sin 65^\circ + \cos 65^\circ = \sin 65^\circ + \sin 25^\circ$
 $= 2 \sin 45^\circ \cos 20^\circ = \sqrt{2} \cos 20^\circ.$

15. First side = $2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ$
 $= \cos 20^\circ - \cos 20^\circ = 0.$

16. First side = $2 \cos 48^\circ \sin 30^\circ + \cos (180^\circ - 48^\circ)$
 $= \cos 48^\circ - \cos 48^\circ = 0.$

17. $\sin^2 5A - \sin^2 2A = \sin (5A + 2A) \sin (5A - 2A) = \sin 7A \sin 3A.$

18. $\cos 2A \cos 5A = \frac{1}{2} (\cos 7A + \cos 3A) = \frac{1}{2} \left(2 \cos^2 \frac{7A}{2} - 1 + 1 - 2 \sin^2 \frac{3A}{2} \right)$
 $= \cos^2 \frac{7A}{2} - \sin^2 \frac{3A}{2}.$

19. First side = $2 \sin \alpha \cos (\beta + \gamma) + 2 \sin \alpha \cos (\beta - \gamma)$
 $= 2 \sin \alpha \{ \cos (\beta + \gamma) + \cos (\beta - \gamma) \}$
 $= 4 \sin \alpha \cos \beta \cos \gamma.$

20. First side = $2 \sin \gamma \sin (\alpha - \beta) + 2 \sin (\alpha + \beta) \sin \gamma$
 $= 2 \sin \gamma \{ \sin (\alpha - \beta) + \sin (\alpha + \beta) \}$
 $= 4 \sin \alpha \cos \beta \sin \gamma.$

21. First side = $2 \sin (\alpha + \beta) \cos (\alpha - \beta) - 2 \sin (\alpha + \beta) \cos (\alpha + \beta + 2\gamma)$
 $= 2 \sin (\alpha + \beta) \{ \cos (\alpha - \beta) - \cos (\alpha + \beta + 2\gamma) \}$
 $= 4 \sin (\alpha + \beta) \sin (\beta + \gamma) \sin (\gamma + \alpha).$

22. First side = $2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \cos \frac{\alpha + \beta}{2}$
 $= 2 \cos \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta + 2\gamma}{2} \right)$
 $= 4 \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2} \cos \frac{\alpha + \beta}{2}.$

23. First side = $2 \sin A \{ \cos 2A - \cos 120^\circ \}$
 $= 2 \sin A \left\{ \cos 2A + \frac{1}{2} \right\}$
 $= 2 \sin A \cos 2A + \sin A$
 $= \sin 3A - \sin A + \sin A = \sin 3A.$

24. First side = $2 \cos \theta \left\{ \cos \frac{4\pi}{3} + \cos 2\theta \right\} = 2 \cos \theta \left(-\frac{1}{2} + \cos 2\theta \right)$
 $= -\cos \theta + 2 \cos \theta \cos 2\theta$
 $= -\cos \theta + \cos 3\theta + \cos \theta$
 $= \cos 3\theta.$

25. First side = $\cos \theta + 2 \cos \frac{2\pi}{3} \cos \theta$
 $= \cos \theta - \cos \theta = 0.$

26. First side = $\frac{1}{2} \{ 1 + \cos 2A + 1 + \cos 2(60^\circ + A) + 1 + \cos 2(60^\circ - A) \}$
 $= \frac{1}{2} \{ 3 + \cos 2A + 2 \cos 120^\circ \cos 2A \}$
 $= \frac{1}{2} \{ 3 + \cos 2A - \cos 2A \} = \frac{3}{2}.$

27. First side = $\frac{1}{2} \{ 3 - \cos 2A - \cos 2(120^\circ + A) - \cos 2(120^\circ - A) \}$
 $= \frac{1}{2} \{ 3 - \cos 2A - 2 \cos 240^\circ \cos 2A \}$
 $= \frac{1}{2} \{ 3 - \cos 2A + \cos 2A \} = \frac{3}{2}.$

$$\begin{aligned}
 28. \quad & \cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{2} \cos 20^\circ (\cos 120^\circ + \cos 40^\circ) \\
 & = \frac{1}{2} \cos 20^\circ \left(-\frac{3}{2} + 2 \cos^2 20^\circ \right) \\
 & = \frac{1}{4} (4 \cos^3 20^\circ - 3 \cos 20^\circ) = \frac{1}{4} \cos 60^\circ = \frac{1}{8}.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{2} \sin 20^\circ \{ \cos 40^\circ - \cos 120^\circ \} \\
 & = \frac{1}{2} \sin 20^\circ \left\{ \frac{3}{2} - 2 \sin^2 20^\circ \right\} \\
 & = \frac{1}{4} \sin 60^\circ = \frac{\sqrt{3}}{8}.
 \end{aligned}$$

EXAMPLES. XII. d. PAGE 119.

$$\begin{aligned}
 1. \quad & \text{First side} = 2 \cos(A+B) \sin(A-B) + 2 \sin C \cos C \\
 & = -2 \cos C \sin(A-B) + 2 \sin(A+B) \cos C \\
 & = 2 \cos C \{ \sin(A+B) - \sin(A-B) \} \\
 & = 4 \cos A \sin B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad & \text{First side} = 2 \cos(A+B) \sin(A-B) - 2 \sin C \cos C \\
 & = -2 \cos C \{ \sin(A-B) + \sin(A+B) \} \\
 & = -4 \sin A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & \text{First side} = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 & = 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right\} \\
 & = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & \text{First side} = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 & = 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right\} \\
 & = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & \text{First side} = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2} + 2 \cos^2 \frac{C}{2} - 1 \\
 & = 2 \cos \frac{C}{2} \left\{ \sin \frac{B-A}{2} + \sin \frac{B+A}{2} \right\} - 1 \\
 & = 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.
 \end{aligned}$$

$$6. \text{ First side} = \frac{4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$7. \text{ We have } \tan \frac{A+B}{2} \tan \frac{C}{2} = 1;$$

$$\therefore \frac{\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = 1.$$

Multiply up and transpose and we obtain the required result.

$$8. \text{ First side} = \frac{2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \cos^2 \frac{A}{2} - 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}}$$

$$= \frac{\sin \frac{B+C}{2} + \sin \frac{B-C}{2}}{\sin \frac{B+C}{2} - \sin \frac{B-C}{2}} = \frac{2 \sin \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \sin \frac{C}{2}} = \tan \frac{B}{2} \cot \frac{C}{2}.$$

$$9. \text{ First side} = 2 \cos(A+B) \cos(A-B) + 2 \cos^2 C + 4 \cos A \cos B \cos C$$

$$= -2 \cos C \{\cos(A-B) + \cos(A+B)\} + 4 \cos A \cos B \cos C$$

$$= -4 \cos A \cos B \cos C + 4 \cos A \cos B \cos C = 0.$$

$$10. \text{ We have } \cot(A+B) = \cot(180^\circ - C) = -\cot C;$$

$$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C;$$

whence by multiplying up and transposing we obtain the required result.

$$11. \text{ First side} = \frac{\sin(B+C)}{\sin B \sin C} \cdot \frac{\sin(C+A)}{\sin C \sin A} \cdot \frac{\sin(A+B)}{\sin A \sin B}$$

$$= \frac{\sin A \sin B \sin C}{\sin^2 A \sin^2 B \sin^2 C} = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

$$12. \text{ First side} = \frac{1}{2} \{3 + \cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C\}$$

$$= \frac{1}{2} (3 - 1), \text{ by Example 9,}$$

$$= 1.$$

13. First side = $\frac{1}{2}(3 - \cos A - \cos B - \cos C)$

$$= \frac{1}{2} \left(3 - 1 - 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \quad [\text{See Ex. 3, p. 119.}]$$

$$= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

14. First side = $\frac{1}{2} \{2 + 2 \cos^2 2A + \cos 4B + \cos 4C\}$

$$= \frac{1}{2} \{2 + 2 \cos^2 2A + 2 \cos 2(B+C) \cos 2(B-C)\}$$

$$= 1 + \cos 2A \{\cos 2(B+C) + \cos 2(B-C)\}$$

$$= 1 + 2 \cos 2A \cos 2B \cos 2C.$$

15. First side = $\frac{\cot B + \cot C}{\frac{1}{\cot B} + \frac{1}{\cot C}} + \frac{\cot C + \cot A}{\frac{1}{\cot C} + \frac{1}{\cot A}} + \frac{\cot A + \cot B}{\frac{1}{\cot A} + \frac{1}{\cot B}}$

$$= \cot B \cot C + \cot C \cot A + \cot A \cot B$$

$$= 1. \quad [\text{See Ex. 10.}]$$

16. First side = $\frac{\tan A \tan B \tan C}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$

$$= \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \cdot \frac{1}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot 2 \sin \frac{B}{2} \cos \frac{B}{2} \cdot 2 \sin \frac{C}{2} \cos \frac{C}{2}}{\cos A \cos B \cos C \cdot 16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$= \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}.$$

EXAMPLES. XII. e. PAGE 121.

1. First side = $\frac{1}{2} \{\sin 2\alpha - \sin 2\beta + \sin 2\beta - \sin 2\gamma + \sin 2\gamma - \sin 2\delta + \sin 2\delta - \sin 2\alpha\} = 0.$

2. First side = $\cot \gamma - \cot \beta + \cot \alpha - \cot \gamma + \cot \beta - \cot \alpha = 0.$

[See Ex. XI. a, 10.]

$$\begin{aligned}
 3. \text{ First side} &= \frac{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}} \\
 &= \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ First side} &= \sin \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \sin \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \cos \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \cos \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ First side} &= \cos \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \cos \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \sin \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 6. \text{ First side} &= \cos A \cos 2A + \sin A \sin 2A - (\sin A \cos 2A + \cos A \sin 2A) \\
 &= \cos(2A - A) - \sin(2A + A) \\
 &= \cos A - \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 7. a \cos 2\theta + b \sin 2\theta &= \frac{a(1 - \tan^2 \theta)}{1 + \tan^2 \theta} + \frac{2b \tan \theta}{1 + \tan^2 \theta}, \quad [\text{Art. 124.}] \\
 &= \frac{a(a^2 - b^2)}{a^2 + b^2} + \frac{2ab^2}{a^2 + b^2} = a \cdot \frac{a^2 + b^2}{a^2 + b^2} = a.
 \end{aligned}$$

$$8. \sin 2A + \cos 2A = \frac{2 \tan A}{1 + \tan^2 A} + \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{(1 + \tan A)^2 - 2 \tan^2 A}{1 + \tan^2 A}.$$

$$9. \sin 4A = 2 \sin 2A \cos 2A = \frac{4 \tan A (1 - \tan^2 A)}{(1 + \tan^2 A)^2}.$$

10. We have $A + B = 45^\circ$;

$$\therefore \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1;$$

$$\therefore \tan A + \tan B + \tan A \tan B = 1.$$

$$\therefore (1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2.$$

$$\begin{aligned}
 11. \text{ First side} &= \frac{\cos(15^\circ - A) \cos(15^\circ + A) + \sin(15^\circ - A) \sin(15^\circ + A)}{\sin(15^\circ - A) \cos(15^\circ + A)} \\
 &= \frac{2 \cos 2A}{\sin 30^\circ - \sin 2A} = \frac{4 \cos 2A}{1 - 2 \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 12. \text{ First side} &= \frac{\cos^2(15^\circ + A) + \sin^2(15^\circ + A)}{\sin(15^\circ + A) \cos(15^\circ + A)} = \frac{2}{\sin(30^\circ + 2A)} \\
 &= \frac{2}{\frac{1}{2} \cos 2A + \frac{\sqrt{3}}{2} \sin 2A} = \frac{4}{\cos 2A + \sqrt{3} \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 13. \text{ First side} &= \frac{\sin(A+30^\circ)\sin(A-30^\circ)}{\cos(A+30^\circ)\cos(A-30^\circ)} \\
 &= \frac{\cos 60^\circ - \cos 2A}{\cos 60^\circ + \cos 2A} = \frac{1 - 2 \cos 2A}{1 + 2 \cos 2A}.
 \end{aligned}$$

$$\begin{aligned}
 14. \text{ First side} &= (4 \cos^2 A - 1)(2 \cos 2A - 1) \\
 &= (2 \cos 2A + 1)(2 \cos 2A - 1) \\
 &= 4 \cos^2 2A - 1 = 2 \cos 4A + 1.
 \end{aligned}$$

$$15. \text{ We have } \tan(\theta + \phi + \psi) = \frac{\tan \theta + \tan \phi + \tan \psi - \tan \theta \tan \phi \tan \psi}{1 - \tan \theta \tan \phi - \tan \phi \tan \psi - \tan \psi \tan \theta}, \\
 \therefore \text{ if } \theta + \phi + \psi = 0,$$

we have $\tan \theta + \tan \phi + \tan \psi = \tan \theta \tan \phi \tan \psi$.

Now putting $\theta = \beta - \gamma$, $\phi = \gamma - \alpha$, $\psi = \alpha - \beta$,

we have at once the required identity.

16. First side

$$\begin{aligned}
 &= 2 \sin \frac{\beta - \alpha}{2} \cos \frac{\alpha + \beta - 2\gamma}{2} + \sin(\alpha - \beta) + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -2 \sin \frac{\alpha - \beta}{2} \left\{ \cos \frac{\alpha + \beta - 2\gamma}{2} - \cos \frac{\alpha - \beta}{2} \right\} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -4 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha - \gamma}{2} \sin \frac{\gamma - \beta}{2} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 17. \text{ First side} &= \frac{1}{2} \{ \cos 2(\beta - \gamma) + \cos 2(\gamma - \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \frac{1}{2} \{ 2 \cos(\alpha - \beta) \cos(\beta - 2\gamma + \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \cos(\alpha - \beta) [\cos(\beta - 2\gamma + \alpha) + \cos(\alpha - \beta)] + 1 \\
 &= 1 + 2 \cos(\beta - \gamma) \cos(\gamma - \alpha) \cos(\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 18. \text{ First side} &= 1 + \frac{1}{2} (\cos 2\alpha + \cos 2\beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \{ \sin \alpha \sin \beta - \cos \alpha \cos \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

$$\begin{aligned}
 19. \text{ First side} &= 1 - \frac{1}{2} (\cos 2\alpha + \cos 2\beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta) \\
 &= 1 - \cos(\alpha + \beta) \{ \cos(\alpha - \beta) - 2 \sin \alpha \sin \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

20. First side = $\cos 12^\circ + 2 \cos 72^\circ \cos 12^\circ$

$$= \cos 12^\circ \left(1 + \frac{\sqrt{5}-1}{2} \right) = \cos 12^\circ \left(\frac{\sqrt{5}+1}{2} \right)$$

$$= 2 \cos 12^\circ \cos 36^\circ = \cos 24^\circ + \cos 48^\circ.$$

21. $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$ [Ex. 2, p. 121.]

$$= 4 \cos \frac{B+C}{4} \cos \frac{C+A}{4} \cos \frac{A+B}{4}.$$

22. Second side = $2 \cos \frac{\pi - B}{4} \left[\cos \frac{2\pi + A + C}{4} + \cos \frac{A - C}{4} \right]$

$$= 2 \cos \frac{A+C}{4} \cos \left(\frac{\pi}{2} + \frac{A+C}{4} \right) + 2 \cos \frac{A+C}{4} \cos \frac{A-C}{4}$$

$$= -2 \cos \frac{A+C}{4} \sin \frac{A+C}{4} + \cos \frac{A}{2} + \cos \frac{C}{2}$$

$$= -\sin \frac{A+C}{2} + \cos \frac{A}{2} + \cos \frac{C}{2}$$

$$= \cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2}.$$

23. This may be done in the same way as the two preceding examples. The following solution exhibits another method.

From Example 3 of Art. 135,

if $\alpha + \beta + \gamma = \pi$, then $\cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$.

Put $\alpha = \frac{\pi}{2} - \frac{A}{2}$, $\beta = \frac{\pi}{2} - \frac{B}{2}$, $\gamma = \frac{\pi}{2} - \frac{C}{2}$;

then $\alpha + \beta + \gamma = \pi$, and after substituting for α, β, γ ,

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 + 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}.$$

24. First side = $\frac{2 \sin(\alpha + \beta) \cos(\alpha - \beta) + \sin 2\gamma}{2 \sin(\alpha + \beta) \cos(\alpha - \beta) + \sin 2\gamma} = \frac{\cos(\alpha - \beta) + \sin \gamma}{\cos(\alpha - \beta) + \sin \gamma}$

$$= \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{\cos(\alpha - \beta) - \cos(\alpha + \beta)} = \cot \alpha \cot \beta.$$

25. Here $\tan(\alpha + \beta) = \tan\left(\frac{\pi}{2} - \gamma\right) = \cot \gamma$;

$$\therefore \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1}{\tan \gamma}.$$

Multiply up, and transpose.

EXAMPLES. XII. f. PAGE 122 A.

9. Use the formulæ of Arts. 123, 124.

$$10. \frac{\sin 3A}{\sin 2A - \sin A} = \frac{3 \sin A - 4 \sin^3 A}{2 \sin A \cos A - \sin A}$$

$$= \frac{3 - 4 \sin^2 A}{2 \cos A - 1} = \frac{4 \cos^2 A - 1}{2 \cos A - 1} = 2 \cos A + 1.$$

$$11. \text{ Here } \cos \alpha = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}. \quad \sin \beta = \sqrt{1 - \frac{3^2}{5^2}} = \frac{4}{5}.$$

$$\therefore \cos(\alpha + \beta) = \frac{24}{25} \cdot \frac{6}{10} - \frac{7}{25} \cdot \frac{8}{10} = \frac{88}{250} = .352;$$

whence, by the Tables, $\alpha + \beta = 69^\circ 23'$.

Again, $\sin \alpha = .28$ gives $\alpha = 16^\circ 15'$,
 $\cos \beta = .6$ gives $\beta = 53^\circ 8'$;
 $\therefore \alpha + \beta = 69^\circ 23'$.

$$12. \sin(\alpha + \beta + \gamma) = \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \sin \alpha \cos(\beta + \gamma),$$

$$\sin(\alpha + \beta - \gamma) + \sin(\alpha - \beta + \gamma) = 2 \sin \alpha \cos(\beta - \gamma);$$

$$\therefore, \text{ by addition, first side} = 2 \sin \alpha \{ \cos(\beta + \gamma) + \cos(\beta - \gamma) \}$$

$$= 4 \sin \alpha \cos \beta \cos \gamma.$$

$$13. \cos 57^\circ + \sin 27^\circ = \cos 57^\circ + \cos 63^\circ$$

$$= 2 \cos 60^\circ \cos 3^\circ$$

$$= \cos 3^\circ.$$

Again, $\cos 57^\circ = .5446$,
 $\cos 63^\circ = .4540$;
 $\therefore \cos 57^\circ + \cos 63^\circ = .9986 = \cos 3^\circ$, by the Tables.

$$14. \text{ Expression} = 2 \sin 5\alpha (\cos 5\alpha + \cos \alpha)$$

$$= \sin 10\alpha + \sin 6\alpha + \sin 4\alpha.$$

$$15. \text{ Expression} = 2 \cos 20^\circ (\cos 70^\circ + \cos 10^\circ)$$

$$= \cos 90^\circ + \cos 50^\circ + \cos 30^\circ + \cos 10^\circ$$

$$= \cos 50^\circ + \cos 30^\circ + \cos 10^\circ$$

$$= .6428 + .8660 + .9848$$

$$= 2.4936.$$

16. See MISCELLANEOUS EXAMPLES K. No. 242.

$$\begin{aligned}
 17. \quad (i) \quad \text{First side} &= x^2 + y^2 + xy (\cot^2 a + \tan^2 a) \\
 &= (x+y)^2 - 2xy + xy \left(\frac{1}{\tan^2 a} + \tan^2 a \right) \\
 &= (x+y)^2 + xy \left(\frac{1 + \tan^4 a}{\tan^2 a} - 2 \right) \\
 &= (x+y)^2 + 4xy \frac{(1 - \tan^2 a)^2}{4 \tan^2 a} \\
 &= (x+y)^2 + 4xy \cot^2 2a.
 \end{aligned}$$

(ii) Use the formula of Ex. 12 on p. 97.

$$\begin{aligned}
 (iii) \quad &(\cos a + \cos 7a) + \cos (3a + \cos 5a) \\
 &= 2 \cos 4a \cos 3a + 2 \cos 4a \cos a \\
 &= 2 \cos 4a (\cos 3a + \cos a) \\
 &= \frac{\sin 8a}{\sin 4a} \cdot 2 \cos 2a \cdot \cos a \\
 &= \frac{\sin 8a \cdot 2 \cos 2a \cos a}{2 \sin 2a \cos 2a} \\
 &= \frac{1}{2} \sin 8a \cdot \frac{2 \cos a}{\sin 2a} = \frac{1}{2} \sin 8a \operatorname{cosec} a.
 \end{aligned}$$

$$18. \quad \sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{6}{10} \cdot \frac{8}{10} = .96,$$

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 2 \cdot \frac{64}{100} - 1 = .28.$$

Again, $\cos \theta = .8$ gives $\theta = 36^\circ 52'$. $\therefore 2\theta = 73^\circ 44'$,
and from the Tables, $\sin 73^\circ 44' = .96$, $\cos 73^\circ 44' = .28$.

$$\begin{aligned}
 19. \quad \text{Here } \cos A &= \cos \left(45^\circ - \frac{B}{2} \right) = \frac{1}{\sqrt{2}} \left(\cos \frac{B}{2} + \sin \frac{B}{2} \right) \\
 &= \frac{1}{\sqrt{2}} \sqrt{\cos^2 \frac{B}{2} + \sin^2 \frac{B}{2} + \sin B} = \sqrt{\frac{1 + \sin B}{2}}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad (i) \quad \text{First side} &= \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ} = \frac{4 \left(\frac{1}{2} \cos 10^\circ - \frac{\sqrt{3}}{2} \sin 10^\circ \right)}{\sin 20^\circ} \\
 &= \frac{4 \sin (30^\circ - 10^\circ)}{\sin 20^\circ} = 4.
 \end{aligned}$$

$$(ii) \quad \text{Second side} = \sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

21. Here $\sin C = \sin B, \cos C = -\cos B.$

$$\therefore \text{Second side} = 2 \cos^2 B = 2(1 - \sin^2 B) = 2(1 - \sin B \sin C).$$

$$\begin{aligned} 22. \quad \cos^2 B + \cos^2 C &= \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\ &= 1 + \frac{1}{2}(\cos 2B + \cos 2C) \\ &= 1 + \cos(B+C) \cos(B-C) \\ &= 1 - \cos A \cos(B-C), \text{ for } \cos(B+C) = -\cos A, \\ &= 1 + \cos^2 A - \cos A \{\cos A + \cos(B-C)\} \\ &= 1 + \cos^2 A - \cos A \{\cos(B-C) - \cos(B+C)\} \\ &= 1 + \cos^2 A - 2 \sin B \sin C \cos A. \end{aligned}$$

23. As in Ex. 22,

$$\begin{aligned} \cos^2 B + \cos^2 C &= 1 + \cos(B+C) \cos(B-C) \\ &= 1 + \cos A \cos(B-C), \text{ for } \cos(B+C) = \cos A, \\ &= 1 + \cos^2 A + \cos A \{\cos(B-C) - \cos(B+C)\} \\ &= 1 + \cos^2 A + 2 \sin B \sin C \cos A. \end{aligned}$$

$$\begin{aligned} 24. \quad \cos^2 A \cos 2B - \cos^2 B \cos 2A &= \cos^2 A (\cos^2 B - \sin^2 B) \\ &\quad - \cos^2 B (\cos^2 A - \sin^2 A) \\ &= \cos^2 B \sin^2 A - \cos^2 A \sin^2 B \\ &= (1 - \sin^2 B) \sin^2 A - (1 - \sin^2 A) \sin^2 B \\ &= \sin^2 A - \sin^2 B; \end{aligned}$$

whence, by transposition, we have the required result.

$$\begin{aligned} 25. \quad \tan 50^\circ - \tan 40^\circ &= \frac{\sin 50^\circ}{\cos 50^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ} \\ &= \frac{\sin 50^\circ \cos 40^\circ - \cos 50^\circ \sin 40^\circ}{\cos 50^\circ \cos 40^\circ} \\ &= \frac{\sin(50^\circ - 40^\circ)}{\cos 50^\circ \cos 40^\circ} = \frac{\sin 10^\circ}{\cos 50^\circ \sin 50^\circ} \\ &= \frac{2 \sin 10^\circ}{\sin 100^\circ} = \frac{2 \sin 10^\circ}{\cos 10^\circ} = 2 \tan 10^\circ. \end{aligned}$$

26. Here $\tan \theta = 2$; $\therefore \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{4}{1+4} = \frac{4}{5} = .8$

and $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - 4}{1+4} = -\frac{3}{5} = -.6$.

Again, from the Tables, $\cot \theta = .5$ gives $\theta = 63^\circ 26'$.

$$\therefore 2\theta = 126^\circ 52'.$$

$$\therefore \sin 2\theta = \sin (180^\circ - 126^\circ 52') = \sin 53^\circ 8'$$

$= .8$, from the Tables.

$$\cos 2\theta = -\cos 53^\circ 8' = -.6, \text{ from the Tables.}$$

27. If $\tan \alpha = .362$, the equation may be written

$$\tan \alpha \cos \theta + \sin \theta = 1, \text{ or } \sin \alpha \cos \theta + \cos \alpha \sin \theta = \cos \alpha.$$

$$\therefore \sin (\alpha + \theta) = \cos \alpha = \sin (90^\circ - \alpha).$$

Now from the Tables, $\alpha = 19^\circ 54'$.

$$\therefore \sin (19^\circ 54' + \theta) = \sin (90^\circ - 19^\circ 54') = \sin 70^\circ 6'.$$

$$\therefore 19^\circ 54' + \theta = 70^\circ 6', \text{ or } 180^\circ - 70^\circ 6';$$

whence

$$\theta = 50^\circ 12', \text{ or } 90^\circ.$$

EXAMPLES. XIII. a. PAGE 128.

1. $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{225 + 49 - 169}{210} = \frac{105}{210} = \frac{1}{2}; \therefore C = 60^\circ.$

2. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 25 - 49}{30} = -\frac{15}{30} = -\frac{1}{2}; \therefore A = 120^\circ.$

3. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 (3 + 1 - 1)}{5^2 2 \sqrt{3}} = \frac{\sqrt{3}}{2}; \therefore A = 30^\circ.$

Also $a = c$; $\therefore C = A = 30^\circ$; hence $B = 180^\circ - A - C = 120^\circ$.

4. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{961 + 98 - 625}{434 \sqrt{2}} = \frac{1}{\sqrt{2}}; \therefore A = 45^\circ.$

5. Let $a=2$, $b=2\frac{2}{3}$, $c=3\frac{1}{3}$.

$$\text{Then } \cos C = \frac{4 + \frac{64}{9} - \frac{100}{9}}{\frac{32}{3}} = 0; \therefore C=90^\circ.$$

$$6. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 6 - (4 + 2\sqrt{3})}{4\sqrt{6}} = \frac{3 - \sqrt{3}}{2\sqrt{6}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \therefore A=75^\circ;$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{6 + 4 - 2\sqrt{3} - 4}{2\sqrt{6}(\sqrt{3}+1)} = \frac{3 + \sqrt{3}}{\sqrt{2}(3 + \sqrt{3})} = \frac{1}{\sqrt{2}}; \therefore B=45^\circ;$$

$$\therefore C=180^\circ - 75^\circ - 45^\circ = 60^\circ.$$

$$7. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 2\sqrt{3} - 2}{4(\sqrt{3}-1)} = \frac{3 - \sqrt{3}}{2(\sqrt{3}-1)} = \frac{\sqrt{3}}{2}; \therefore A=30^\circ;$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{4 - 2\sqrt{3} + 2 - 4}{2\sqrt{2}(\sqrt{3}-1)} = \frac{1 - \sqrt{3}}{\sqrt{2}(\sqrt{3}-1)} = -\frac{1}{\sqrt{2}};$$

$$\therefore B=135^\circ;$$

$$\therefore C=180^\circ - 30^\circ - 135^\circ = 15^\circ.$$

$$8. \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 25 - 19}{80} = \frac{70}{80} = .875; \therefore C=28^\circ 57'.$$

$$9. \cos C = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5} = \frac{16 + 25 - 49}{40} = -\frac{8}{40} = -\frac{1}{5} = -\cos 78^\circ 28';$$

$$\therefore C=180^\circ - 78^\circ 28'=101^\circ 32'.$$

$$10. c^2 = a^2 + b^2 - 2ab \cos C = 4 + 4 + 2\sqrt{3} - 2 \cdot 2(\sqrt{3}+1)\frac{1}{2} = 6.$$

$$11. b^2 = c^2 + a^2 - 2ca \cos B = 9 + 25 - 2 \cdot 3 \cdot 5 \left(-\frac{1}{2} \right) = 9 + 25 + 15 = 49;$$

$$\therefore b=7.$$

$$12. a^2 = b^2 + c^2 - 2bc \cos A = 49 + 36 - 2 \cdot 7 \cdot 6 \times .2501$$

$$= 49 + 36 - 21.0042 = 64, \text{ approx.}; \text{ whence } a=8.$$

$$13. a^2 = b^2 + c^2 - 2bc \cos A = 64 + 121 - 2 \cdot 8 \cdot 11 \left(-\frac{1}{16} \right)$$

$$= 64 + 121 + 11 = 196;$$

$$\therefore a=14.$$

14. $b^2 = c^2 + a^2 - 2ca \cos B = 9 + 49 - 2 \cdot 3 \cdot 7 (-\cdot 5476) = 81$, approx.;

$$\therefore b = 9.$$

15. $b^2 = c^2 + a^2 - 2ca \cos B = 48 - 24\sqrt{3} + 24 - 2 \cdot 2\sqrt{6} \cdot 2\sqrt{3} (\sqrt{3} - 1) \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$$= 48 - 24\sqrt{3} + 24 - 4 \cdot 3 \cdot (4 - 2\sqrt{3}) = 24;$$

$$\therefore b = 2\sqrt{6}; \text{ whence } A = B = 75^\circ, \text{ and } C = 30^\circ.$$

16. $a^2 = b^2 + c^2 - 2bc \cos A = 4 + 6 + 2\sqrt{5} - 2 \cdot 2(\sqrt{5} + 1) \frac{\sqrt{5} - 1}{4}$

$$= 4 + 6 + 2\sqrt{5} - 4 = 6 + 2\sqrt{5};$$

$$\therefore a = \sqrt{5} + 1; \text{ whence } C = A = 72^\circ, \text{ and } B = 36^\circ.$$

17. $C = 75^\circ$; whence $a = c$.

Also $a = \frac{b \sin A}{\sin B} = \frac{\sqrt{8}(\sqrt{3} + 1)2}{2\sqrt{2}} = 2\sqrt{3} + 2$.

18. $c = \frac{b \sin C}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} - 1$.

Also $A = 105^\circ$; whence $a = \frac{b \sin A}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} + 1$.

19. $C = 30^\circ$; whence $a = \frac{c \sin A}{\sin C} = \frac{\sqrt{2} \times 2}{\sqrt{2}} = 2$;

$$b = \frac{a \sin B}{\sin A} = 2 \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \sqrt{2} = \sqrt{3} + 1.$$

20. Here $\frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin 75^\circ}{\sin 45^\circ} = \frac{\sqrt{3} + 1}{2\sqrt{2}} / \frac{1}{\sqrt{2}} = \frac{\sqrt{3} + 1}{2}$.

21. $\sin A = \frac{a \sin C}{c} = \frac{2}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}$;

$$\therefore b = a \cos C + c \cos A = 2 \left(-\frac{1}{2} \right) + 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3 - 1 = 2.$$

$$22. \sin A = \frac{a \sin B}{b} = \frac{3}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2};$$

$$\therefore c = a \cos B + b \cos A = 3 \cdot \frac{1}{2} + 3\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6.$$

23. We have $(b+c)^2 - a^2 = 3bc$;

$$\therefore \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}; \text{ whence } A = 60^\circ.$$

$$24. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{12 + 6 - (12 + 6\sqrt{3})}{2 \cdot 2\sqrt{3} \cdot \sqrt{6}}$$

$$= \frac{6(1 - \sqrt{3})}{12\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}} = -\cos 75^\circ;$$

$\therefore A = 105^\circ$; similarly $C = 30^\circ$.

$$\therefore \sin B = \frac{b \sin C}{c} = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}}; \text{ whence } B = 45^\circ.$$

25. The sides are proportional to $\sqrt{3} + 1, \sqrt{3} - 1, \sqrt{6}$;

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 - 2\sqrt{3} + 6 - (4 + 2\sqrt{3})}{2\sqrt{6}(\sqrt{3} - 1)}$$

$$= \frac{6 - 4\sqrt{3}}{2\sqrt{6}(\sqrt{3} - 1)} = \frac{2\sqrt{3}(\sqrt{3} - 2)}{2\sqrt{3}\cdot\sqrt{2}(\sqrt{3} - 1)} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}} = -\cos 75^\circ;$$

$\therefore A = 105^\circ$.

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6}}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$\therefore C = 60^\circ$, and $B = 15^\circ$.

$$26. \text{ Here } b = \frac{\sqrt{6} + \sqrt{2}}{4}, \quad c = \frac{\sqrt{6} - \sqrt{2}}{4}, \quad A = 60^\circ;$$

$$\therefore a^2 = \frac{(\sqrt{6} + \sqrt{2})^2 + (\sqrt{6} - \sqrt{2})^2}{16} - \frac{2 \cdot 4}{16} \cdot \frac{1}{2} = \frac{12}{16} = \frac{3}{4};$$

$$\therefore a = \frac{\sqrt{3}}{2}.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6} - \sqrt{2}}{4} \cdot \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}};$$

whence

$$C = 15^\circ, \text{ and } B = 105^\circ.$$

EXAMPLES. XIII. b. PAGE 132.

$$1. \sin B = \frac{b \sin A}{a} = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}.$$

$\therefore B=60^\circ$ or 120° ; and since $a < b$, both these values are admissible.
Hence $C=90^\circ$ or 30° .

$$c = \frac{a \sin C}{\sin A} = 2 \text{ or } 1, \text{ on reduction.}$$

$$2. \sin B = \frac{b \sin C}{c} = \frac{3\sqrt{2}}{2\sqrt{3}} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore B=60^\circ$ or 120° ; and since $c < b$, both these values are admissible.
Hence $A=75^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = 3 + \sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$3. \sin A = \frac{a \sin C}{c} = \frac{2}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore A=45^\circ$, the other value being inadmissible, since $c > a$. Hence $B=75^\circ$.

$$b = \frac{a \sin B}{\sin A} = \sqrt{3} + 1.$$

4. $\sin C = \frac{c}{a} \sin A = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}$, which is impossible. Thus there is no triangle with the given parts.

$$5. \sin C = \frac{c}{b} \sin B = \frac{2\sqrt{3}}{\sqrt{6}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore C=45^\circ$ or 135° ; and since $b < c$, both values are admissible. Hence $A=105^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = \sqrt{3}(\sqrt{3} + 1), \text{ or } \sqrt{3}(\sqrt{3} - 1), \text{ on reduction.}$$

$$6. \sin C = \frac{c}{b} \sin B = \frac{3 + \sqrt{3}}{3\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore C=75^\circ$ or 105° ; and since $b < c$, both values are admissible. Hence $A=45^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = 2\sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$7. \sin A = \frac{a}{c} \sin C = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore A=75^\circ$ or 105° ; and since $c < a$, both values are admissible. Hence $B=90^\circ$ or 60° .

$$b = \frac{c \sin B}{\sin C} = 2\sqrt{6}, \text{ or } 3\sqrt{2}, \text{ on reduction.}$$

$$8. \sin B = \frac{b}{a} \sin A = \frac{\frac{4(\sqrt{5}+1)}{4}}{\frac{\sqrt{5}-1}{4}} = 1.$$

$\therefore B = 90^\circ$, and there is no ambiguity.

$$c^2 = b^2 - a^2 = (b+a)(b-a) = (8 + \sqrt{80})\sqrt{80}$$

$$= 4(2 + \sqrt{5}) \cdot 4\sqrt{5} = 16\sqrt{5}(2 + \sqrt{5})$$

$$\therefore c = 4\sqrt{5 + 2\sqrt{5}}.$$

$$9. \sin A = \frac{a}{b} \sin B = \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore A = 60^\circ$ or 120° ; but neither of these values is admissible as in each case the sum of the angles would be greater than 180° . Thus the triangle is impossible.

EXAMPLES. XIII. c. PAGE 134.

1. Follows at once from Art. 137.

2. The first side = $b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2$.

3. The first side = $\frac{a^2 + b^2 - c^2}{2} - \frac{a^2 + c^2 - b^2}{2} = b^2 - c^2$.

4. The first side = $(b \cos A + a \cos B) + (c \cos B + b \cos C)$
 $+ (c \cos A + a \cos C) = c + a + b$.

5. The first side = $a(1 - \cos C) + c(1 - \cos A)$
 $= a + c - (a \cos C + c \cos A)$
 $= a + c - b$.

6. The second side = $\frac{a \cos B}{a \cos C} = \frac{\cos B}{\cos C}$.

7. The second side = $\frac{a \sin C}{c \cos A} = \frac{c \sin A}{c \cos A} = \tan A$.

8. Put $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; then

$$\text{the first side} = k(\sin B + \sin C) \sin \frac{A}{2}$$

$$= 2k \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2}$$

$$= 2k \sin \frac{A}{2} \cos \frac{A}{2} \cdot \cos \frac{B-C}{2}$$

$$= k \sin A \cdot \cos \frac{B-C}{2}$$

$$= a \cos \frac{B-C}{2}.$$

$$\begin{aligned}
 9. \quad \text{The first side} &= \frac{k(\sin A + \sin B)}{k \sin C} \sin^2 \frac{C}{2} \\
 &= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} \sin^2 \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \sin \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \cos \frac{A+B}{2} \\
 &= \frac{\cos A + \cos B}{2}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \text{The first side} &= k \{ \sin A \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin (B+C) \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin^2 B - \sin^2 C + \dots + \dots \} = 0.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \text{The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C} \\
 &= \frac{\sin (A+B) \sin (A-B)}{\sin (A+B) \sin (A+B)} = \frac{\sin (A-B)}{\sin (A+B)}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \text{The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C - \sin^2 A} = \frac{\sin (A+B) \sin (A-B)}{\sin (C+A) \sin (C-A)} \\
 &= \frac{\sin C \sin (A-B)}{\sin B \sin (C-A)} = \frac{c \sin (A-B)}{b \sin (C-A)}.
 \end{aligned}$$

EXAMPLES. XIII. d. PAGE 136.

1. Let ABC be the triangle, in which $B=C=2A$, and $a=2$;
then $B+C+A=5A=180^\circ$; $\therefore A=36^\circ$, $B=C=72^\circ$;

and $b=c=\frac{a}{\sin A} \sin B = \frac{a \sin 2A}{\sin A} = 2a \cos A = 4 \cdot \frac{\sqrt{5}+1}{4} = \sqrt{5}+1$.

2. $A=180^\circ - B - C=60^\circ$:

If AD be the perpendicular, then $c=AD \operatorname{cosec} 45^\circ=3\sqrt{2}$,

$$b=AD \operatorname{cosec} 75^\circ=\frac{6\sqrt{2}}{\sqrt{3}+1}=3(\sqrt{6}-\sqrt{2})$$

$$a=BD+DC=AD \cot 45^\circ+AD \cot 75^\circ=3+3(2-\sqrt{3})=9-3\sqrt{3}.$$

$$3. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{28 - 16\sqrt{3} + 24 - 12\sqrt{3} - 4}{4\sqrt{2}(9 - 5\sqrt{3})} = \frac{48 - 28\sqrt{3}}{4\sqrt{2}(9 - 5\sqrt{3})}$$

$$= \frac{(12 - 7\sqrt{3})(9 + 5\sqrt{3})}{\sqrt{2} \cdot 6} = \frac{3 - 3\sqrt{3}}{6\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}};$$

$$\therefore A = 180^\circ - 75^\circ = 105^\circ.$$

$$\sin C = \frac{c}{a} \sin A = \frac{3\sqrt{2} - \sqrt{6}}{2} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \cdot \frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{2} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ;$$

and

$$B = 180^\circ - 105^\circ - 60^\circ = 15^\circ.$$

4. We have

$$b - a = 2, ab = 4;$$

$$\therefore b + a = 2\sqrt{5}, \text{ rejecting the negative sign,}$$

$$\therefore a = \sqrt{5} - 1, b = \sqrt{5} + 1;$$

$$\therefore \sin B = \frac{b \sin A}{a} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \cdot \frac{\sqrt{5} - 1}{4} = \frac{\sqrt{5} + 1}{4};$$

$$\therefore B = 54^\circ \text{ or } 126^\circ;$$

$$\therefore C = 108^\circ \text{ or } 36^\circ.$$

$$5. \sin c = \frac{c \sin B}{b} = 150 \cdot \frac{1}{2} \cdot \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ \text{ or } 120^\circ; \text{ both values being admissible since } b < c;$$

$$\therefore A = 90^\circ \text{ or } 30^\circ.$$

$$\text{In the first case, } a = \sin A \cdot \frac{b}{\sin B} = 50\sqrt{3} \times 2 = 100\sqrt{3}.$$

$$\text{In the second case, } a = b = 50\sqrt{3}.$$

$$\text{If } B = 30^\circ, C = 150^\circ, b = 75^\circ;$$

$$\sin C = \frac{150}{75} \times \sin 30^\circ = 1;$$

$$\therefore C = 90^\circ, \text{ and there is no ambiguity.}$$

$$6. \text{ We have at once from a figure, } \sin B = \frac{\sqrt{5} - 1}{4}; \text{ whence } B = 18^\circ;$$

$$\therefore C = 180^\circ - 36^\circ - 18^\circ = 126^\circ.$$

7. Let ABC be the triangle, and let $B = 22\frac{1}{2}^\circ$, $C = 112\frac{1}{2}^\circ$, and let AD be the perpendicular from A on BC . Then $A = 180^\circ - 22\frac{1}{2}^\circ - 112\frac{1}{2}^\circ = 45^\circ$; also

$$AD = AB \sin 22\frac{1}{2}^\circ,$$

$$BC = \frac{AB}{\sin 112\frac{1}{2}^\circ} \cdot \sin 45^\circ = \frac{AB}{\sqrt{2} \cos 22\frac{1}{2}^\circ} = \frac{2AB \sin 22\frac{1}{2}^\circ}{\sqrt{2} \sin 45^\circ} = 2AD.$$

That is, the altitude is half the base.

8. $\frac{\sin A}{\sin B} = \frac{a}{b} = 2$; and $\sin A = 2 \sin B = \sin 3B$;

$$\therefore \frac{3 \sin B - 4 \sin^3 B}{\sin B} = 2, \text{ or } 3 - 4 \sin^2 B = 2.$$

$$\therefore \sin B = \frac{1}{2}, \text{ rejecting the negative value.}$$

$$\therefore B = 30^\circ, A = 90^\circ, C = 60^\circ.$$

Also $c = a \sin C = \frac{a \sqrt{3}}{2}$.

9. Let C be the greatest angle, then

$$\begin{aligned} \cos C &= \frac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)} \\ &= \frac{-2x^3 - 7x^2 - 6x}{2(2x+3)(x^2+2x)} = -\frac{(2x+3)(x+2)}{2(2x+3)(x+2)} = -\frac{1}{2}. \end{aligned}$$

Thus the greatest angle is 120° .

10. First side $= b \cos C + c \cos B - a \cos C - c \cos A = a - b$.

Now $\frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}}$

$$= \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}} = \sin \frac{A-B}{2} \operatorname{cosec} \frac{A+B}{2}.$$

11. $\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{\sin B + \sin(A+B)}{\sin A} = \frac{2 \sin \left(B + \frac{A}{2}\right) \cos \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}}.$

$$\therefore (b+c) \sin \frac{A}{2} = a \sin \left(\frac{A}{2} + B\right).$$

12. $\frac{a+b}{b+c} = \frac{\sin A + \sin B}{\sin B + \sin C} = \frac{\sin B+C + \sin B}{\sin B + \sin C} = \frac{2 \sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$

$$= \frac{\sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B-C}{2}}.$$

13. First side = $\frac{1 - \cos(A - B) \cos(A + B)}{1 - \cos(A - C) \cos(A + C)} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)}$
 $= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{a^2 + b^2}{a^2 + c^2}.$

14. We have $c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$;
 $\therefore (c^2 - a^2 + ab + b^2)(c^2 - a^2 - ab + b^2) = 0$;
 $\therefore c^2 = a^2 + ab + b^2$, or $a^2 - ab + b^2$;

But $c^2 = a^2 + b^2 + 2ab \cos C$;
 $\therefore 2 \cos C = 1$, or -1 ;
 $\therefore C = 60^\circ$, or 120° .

15. See figure of the Ambiguous Case on page 131.

(1) $c_1 - c_2 = B_1 B_2 = 2B_1 D = 2a \cos B_1$.

(2) $\cos \frac{C_1 - C_2}{2} = \cos B_1 CD = \frac{CD}{CB_1} = \frac{b \sin A}{a}$.

(3) c_1, c_2 are the roots of the quadratic

$$c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0;$$

[Art. 150.]

$$\therefore c_1 + c_2 = 2b \cos A; \quad c_1 c_2 = b^2 - a^2.$$

$$\begin{aligned} \therefore c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A &= c_1^2 + c_2^2 - 2c_1 c_2 (2 \cos^2 A - 1) \\ &= (c_1 + c_2)^2 - 4(b^2 - a^2) \cos^2 A \\ &= 4b^2 \cos^2 A - 4b^2 \cos^2 A + 4a^2 \cos^2 A \\ &= 4a^2 \cos^2 A. \end{aligned}$$

(4) We have $C_1 + C_2 = 2 \angle ACD$;

$$C_1 - C_2 = 2 \angle B_1 CD;$$

$$\begin{aligned} \therefore \sin \frac{C_1 + C_2}{2} \sin \frac{C_1 - C_2}{2} &= \sin ACD \sin B_1 CD \\ &= \cos CAB_1 \cos CB_1 A \\ &= \cos A \cos B. \end{aligned}$$

16. See figure of Art. 148 (iii). We have $A = 45^\circ$; $\therefore \angle ACD = 45^\circ$.

Hence $DC = DA = \frac{c_1 + c_2}{2}$; also $DB_2 = \frac{c_1 - c_2}{2}$;

$$\therefore CB_2^2 = \left(\frac{c_1 + c_2}{2}\right)^2 + \left(\frac{c_1 - c_2}{2}\right)^2 = \frac{c_1^2 + c_2^2}{2};$$

$$\therefore \cos^2 B_2 CD = \frac{CD^2}{CB_2^2} = \frac{(c_1 + c_2)^2}{2(c_1^2 + c_2^2)};$$

$$\therefore \cos B_1 CB_2 = 2 \cos^2 B_2 CD - 1 = \frac{(c_1 + c_2)^2}{c_1^2 + c_2^2} - 1 = \frac{2c_1 c_2}{c_1^2 + c_2^2}.$$

17. From the given condition we have

$$\sin C \cos A + 2 \cos C \sin C = \sin B \cos A + 2 \cos B \sin B,$$

or $\cos A (\sin C - \sin B) = \sin 2B - \sin 2C.$

This easily reduces to

$$\cos A \sin \frac{B-C}{2} \cos \frac{B+C}{2} = 2 \cos A \sin \frac{B-C}{2} \cos \frac{B-C}{2}.$$

Now $\cos \frac{B+C}{2}$ cannot equal $2 \cos \frac{B-C}{2}$; hence we must have

$$\cos A = 0, \text{ which gives } A = 90^\circ;$$

or $\sin \frac{B-C}{2} = 0, \text{ in which case } B = C.$

18. Since a, b, c are in A.P.; we have $a-b=b-c$;

$$\therefore \sin A - \sin B = \sin B - \sin C;$$

$$\therefore 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2} = 2 \sin \frac{B-C}{2} \cos \frac{B+C}{2};$$

$$\therefore \frac{\sin \frac{A-B}{2}}{\sin \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin \frac{B-C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}};$$

$$\therefore \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2};$$

That is $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

19. Let $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; then

$$\text{first side} = \frac{k^2 \sin A \sin \overline{B+C} \sin \overline{B-C}}{\sin B + \sin C} + \dots + \dots$$

$$= k^2 \left[\frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \dots + \dots \right]$$

$$= k^2 [\sin A (\sin B - \sin C) + \dots + \dots] = 0.$$

MISCELLANEOUS EXAMPLES. D. PAGE 138.

1. (1) $\tan 2\theta \cot \theta - 1 = \frac{2}{1 - \tan^2 \theta} - 1 = \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \sec 2\theta.$

(2) $\sin \alpha - \cot \theta \cos \alpha = \frac{\sin \alpha \sin \theta - \cos \alpha \cos \theta}{\sin \theta} = -\operatorname{cosec} \theta \cos(\alpha + \theta).$

2. $c^2 = a^2 + b^2 - 2ab \cos C = 48^2 + 35^2 - 48 \times 35$
 $= 13^2 + 48 \times 35 = 1849;$
 $\therefore c = 43.$

3. Here $\tan \alpha = \sqrt{\left(\frac{17}{8}\right)^2 - 1} = \frac{15}{8};$
also $\tan \beta = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \frac{8}{15} = \cot \alpha;$
 $\therefore \alpha + \beta = 90^\circ;$
 $\therefore \tan(\alpha + \beta) = \infty,$ and $\operatorname{cosec}(\alpha + \beta) = 1.$

4. The expression $= \frac{2 \sin 8\alpha \cos 15\alpha}{2 \sin 8\alpha \cos 6\alpha} = \frac{\cos 15\alpha}{\cos 6\alpha} = -1,$
since $15\alpha = \pi - 6\alpha.$

5. First side $= \frac{1}{2} (\sin 3\theta - \sin \theta + \sin 5\theta - \sin 3\theta + \sin 7\theta - \sin 5\theta)$
 $= \frac{1}{2} (\sin 7\theta - \sin \theta)$
 $= \sin 3\theta \cos 4\theta.$

6. $a^2 = 2 + 4 + 2\sqrt{3} - 2(\sqrt{3} + 1) = 4;$ whence $a = 2.$

Again $\sin B = \frac{b \sin A}{a} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2};$

$\therefore B = 30^\circ$, or 150° ; but the latter value is inadmissible since c is the greatest side. Therefore $C = 180^\circ - (45^\circ + 30^\circ) = 105^\circ.$

7. (1) $2 \sin^2 36^\circ = 1 - \cos 72^\circ = 1 - \sin 18^\circ$
 $= 1 - \frac{\sqrt{5}-1}{4} = \frac{5-\sqrt{5}}{4} = \sqrt{5} \sin 18^\circ.$

(2) $4 \sin 36^\circ \cos 18^\circ = 2(\sin 54^\circ + \sin 18^\circ)$
 $= \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5}.$

8. $\frac{\sin 3\alpha}{\sin \alpha} + \frac{\cos 3\alpha}{\cos \alpha} = \frac{\sin 3\alpha \cos \alpha + \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha} = \frac{2 \sin 4\alpha}{\sin 2\alpha} = 4 \cos 2\alpha.$

9. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 8 + 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2};$
 $\therefore A = 30^\circ.$

$\therefore B = C = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ.$

10. (1) Each expression easily reduces to $\frac{\sin \alpha}{\sin 3\alpha}$.

$$\begin{aligned}(2) \quad \cos \alpha + \cos 2\alpha + \cos 3\alpha &= 2 \cos \frac{3\alpha}{2} \cos \frac{\alpha}{2} + 2 \cos^2 \frac{3\alpha}{2} - 1 \\&= 2 \cos \frac{3\alpha}{2} \left(\cos \frac{\alpha}{2} + \cos \frac{3\alpha}{2} \right) - 1 \\&= 4 \cos \alpha \cos \frac{\alpha}{2} \cos \frac{3\alpha}{2} - 1.\end{aligned}$$

$$\begin{aligned}11. \quad (1) \quad \text{First side} &= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B \\&= 2b \sin C (b \cos C + c \cos B) \\&= 2ab \sin C = 2bc \sin A.\end{aligned}$$

$$\begin{aligned}(2) \quad \text{First side} &= k (\sin A \sin B - C + \dots + \dots) \\&= k (\sin B + C \sin B - C + \dots + \dots) \\&= k (\sin^2 B - \sin^2 C + \dots + \dots) = 0.\end{aligned}$$

$$\begin{aligned}12. \quad \tan(A+B) &= \tan(360^\circ - C+D); \\&\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = - \frac{\tan C + \tan D}{1 - \tan C \tan D}. \\&\therefore \tan A + \tan B + \tan C + \tan D = \tan C \tan D (\tan A + \tan B) \\&\quad + \tan A \tan B (\tan C + \tan D);\end{aligned}$$

$$\begin{aligned}\text{or } \tan A + \tan B + \tan C + \tan D &= \tan A \tan B \tan C \tan D (\cot A + \cot B + \cot C + \cot D).\end{aligned}$$

EXAMPLES. XIV. a. PAGE 145.

For Examples 1—3 see Arts. 151, 152.

For Examples 4—7 see Arts. 162, 163.

$$8. \quad \log 768 = \log (2^8 \times 3) = 8 \log 2 + \log 3 = 2.8853613.$$

$$9. \quad \log 2352 = \log (2^4 \times 3 \times 7^2) = 4 \log 2 + \log 3 + 2 \log 7 = 3.3714373.$$

$$\begin{aligned}10. \quad \log 35.28 &= \log \left(\frac{2^3 \times 3^2 \times 7^2}{10^2} \right) = 3 \log 2 + 2 (\log 3 + \log 7 - 1) \\&= .90309 + 2 (.4771213 + .845098 - 1) \\&= .90309 + .6444386 = 1.5475286.\end{aligned}$$

$$\begin{aligned}11. \quad \log \sqrt{6804} &= \frac{1}{2} \log (2^2 \times 3^5 \times 7) = \frac{1}{2} (2 \log 2 + 5 \log 3 + \log 7) \\&= \frac{1}{2} (.60206 + 2.3856065 + .845098) \\&= 1.9163822.\end{aligned}$$

$$12. \log \sqrt[5]{.00162} = \frac{1}{5} \log \left(\frac{2 \times 3^4}{10^5} \right) = \frac{1}{5} (\log 2 + 4 \log 3 - 5) \\ = \frac{1}{5} (3.2095152) = 1.441903.$$

$$13. \log .0217 = \log \frac{217 - 21}{9000} = \log \frac{196}{9000} = \log \frac{7^2}{15^2 \times 10} \\ = 2(\log 7 - \log 3 - \log 5) - 1 \\ = 2(8450980 - 1.1760913) - 1 = 2.3380134.$$

$$14. \log \cos 60^\circ = \log \left(\frac{1}{2} \right) = -\log 2 = -0.30103 = 1.69897.$$

$$15. \log \sin^3 60^\circ = 3 \log \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2} \log 3 - 3 \log 2 \\ = 0.7156819 - 0.90309 = 1.8125919.$$

$$16. \log \sqrt[3]{\sec 45^\circ} = \frac{1}{3} \log \sqrt{2} = \frac{1}{6} \log 2 = 0.0501716.$$

$$17. \text{The expression} = \log \left[\left(\frac{15}{8} \right)^2 \times \frac{162}{25} \times \left(\frac{4}{9} \right)^3 \right] \\ = \log \left(\frac{15 \times 15 \times 81 \times 2 \times 64}{64 \times 25 \times 81 \times 9} \right) = \log 2.$$

$$18. \text{The expression} \\ = 16(1 - 2 \log 3) - 4(2 \log 5 - 3 \log 2 - \log 3) - 7(3 \log 2 + 1 - 4 \log 3) \\ = 9 - 9 \log 2 - 8 \log 5 = 1 - \log 2 = 0.69897.$$

$$19. \log x = \frac{1}{7} \log 7 = 0.1207283. \therefore x = 1.320469.$$

$$20. \log x = \frac{1}{3} \log \left(\frac{2^2 \times 3^2 \times 7^2}{10^8} \right) = \frac{1}{3} (2 \log 2 + 2 \log 3 + 2 \log 7 - 8) \\ = \frac{1}{3} (3.2464986 - 8) = 2.4154995. \\ \therefore x = 0.0260315.$$

$$21. \log x = 0.5527899 + 2.5527899 + 1.1842633 = 2.2898431.$$

$$22. \log x = \frac{1}{3} \log \frac{11}{10^5} + 2 \log \frac{11^2}{10^2} + \frac{4}{3} \log \frac{11^3}{10^2} - \log (11^2 \times 10^5) \\ = \frac{1}{3} (\log 11 - 5) + 4 (\log 11 - 1) + \frac{4}{3} (3 \log 11 - 2) - 2 \log 11 - 5. \\ = \frac{19}{3} \log 11 - \frac{40}{3} = \frac{1}{3} (21.7864613) = 7.2621538.$$

23. Since $\left(\frac{21}{20}\right)^{300} = \left(\frac{3 \times 7}{2^2 \times 5}\right)^{300}$;

$$\begin{aligned}\therefore \log \left(\frac{21}{20}\right)^{300} &= 300(\log 3 + \log 7 - 1 - \log 2) \\ &= 396.66579 - 390.30900 = 6.35679; \\ \therefore \left(\frac{21}{20}\right)^{300} &\text{ has 7 digits in its integral part.}\end{aligned}$$

Since $\left(\frac{126}{125}\right)^{1000} = \left(\frac{2 \times 3^2 \times 7}{5^3}\right)^{1000}$;

$$\begin{aligned}\therefore \log \left(\frac{126}{125}\right)^{1000} &= 1000(\log 2 + 2 \log 3 + \log 7 - 3 + 3 \log 2) \\ &= 1000(4 \log 2 + 2 \log 3 + \log 7 - 3) \\ &= 3.4606;\end{aligned}$$

$\therefore \left(\frac{126}{125}\right)^{1000}$ has 4 digits in its integral part.

24. 7^4 is the smallest number whose logarithm has characteristic 4.

7^3 is the smallest number whose logarithm has characteristic 3.

\therefore the required number is $7^4 - 7^3 = 2058$.

EXAMPLES. XIV. b. PAGE 149.

$$\begin{aligned}1. \quad \log x &= \frac{2}{3} \log \left(\frac{147 \times 375}{126 \times 16}\right) = \frac{2}{3} \log \left(\frac{7 \times 5^3}{2^5}\right) \\ &= \frac{2}{3} \log \frac{7 \times 1000}{16 \times 16} = \frac{2}{3}(3 + \log 7 - 8 \log 2) \\ &= \frac{2}{3}(3.8450980 - 2.4082400) = \frac{2}{3}(1.4368580) = .9579053; \\ \therefore x &= 9.076226.\end{aligned}$$

$$\begin{aligned}2. \quad \log x &= \frac{1}{3} \log (3^3 \times 2 \times 7) + \frac{1}{2} \log (3^3 \times 2^2) \\ &\quad - \frac{1}{6} \log (3^4 \times 7 \times 2^4) - \frac{1}{3} \log (2 \times 3^5) \\ &\stackrel{1}{=} \frac{1}{2} \log 3 + \frac{1}{3} \log 2 + \frac{1}{6} \log 7 = .4797536; \\ \therefore x &= 3.01824. \quad \begin{array}{r}1003433 \\ 2385606 \\ 1408497 \\ \hline 4797536\end{array}\end{aligned}$$

$$\begin{aligned}
 3. \quad \log x &= \frac{1}{2} \log (10 \times 2^2 \times 3^3) + \frac{5}{3} \log (2^3 \times 3 \div 100) + \log (10 \times 3^4) \\
 &= \frac{43}{6} \log 3 + 6 \log 2 - \frac{11}{6} = \frac{1}{6} (20 \cdot 5162159 - 11) + 1 \cdot 8061800 \\
 &= 1 \cdot 5860360 + 1 \cdot 8061800 = 3 \cdot 3922160; \\
 &\therefore x = 2467 \cdot 266.
 \end{aligned}$$

4. Let x be the value; then $20^x = 800$.

$$\begin{aligned}
 &\therefore x \log 20 = \log 800; \\
 &\therefore x = \frac{\log 800}{\log 20} = \frac{2+3 \log 2}{1+\log 2} = 2 \cdot 23.
 \end{aligned}$$

5. Here $3^x = 49$, or $x \log 3 = 2 \log 7$.

$$\therefore x = \frac{2 \log 7}{\log 3} = 3 \cdot 54.$$

6. Here $125^x = 4000$, or $x \log 5^3 = \log (2^2 \times 10^3)$.

$$\therefore x = \frac{3+2 \log 2}{3 \log 5} = 1 \cdot 72.$$

7. $\log x = \frac{40}{3} \log \left(\frac{378}{10^5} \right) = \frac{40}{3} (\log 2 + 3 \log 3 + \log 7 - 5) = 33 \cdot 699892$.
 \therefore the number of ciphers is 32.

$$\begin{aligned}
 \log x &= 50 \log \frac{259}{9990} = 50 \log \frac{7}{270} \\
 &= 50 (\log 7 - 3 \log 3 - 1) = 50 (1 \cdot 8450980 - 2 \cdot 4313639) \\
 &= 50 (2 \cdot 4137341) = 80 \cdot 6867050. \\
 &\therefore \text{the number of ciphers is 79.}
 \end{aligned}$$

8. Let x be the base; then $x^3 = 11000$;

$$\begin{aligned}
 &\therefore 3 \log x = \log 11 + 3 = 4 \cdot 0413927; \\
 &\therefore \log x = 1 \cdot 3471309.
 \end{aligned}$$

But $\log 222398 = 5 \cdot 3471309$;
 $\therefore x = 22 \cdot 2398$.

9. Here $(x-1) \log 2 = \log 5 = 1 - \log 2$.
 $\therefore x \log 2 = 1$;
 $\therefore x = \frac{1}{\log 2} = \frac{1}{\log 10/3} = 3 \cdot 32$.

10. Here $(x-4) \log 3 = \log 7$.
 $\therefore x-4 = \frac{\log 7}{\log 3} = 1 \cdot 77$;
 $\therefore x = 5 \cdot 77$.

11. Here $(1-x)\log 5 = (x-3)(\log 2 + \log 3);$
 $\therefore x(\log 2 + \log 3 + \log 5) = 3\log 2 + 3\log 3 + \log 5;$
 $\therefore x = \frac{3 \cdot 0334239}{1 \cdot 4771213} = 2 \cdot 05.$

12. Put $\log 2 = a, \log 5 = b;$ then
 $bx = -ay, b(2+y) = a(2-x).$

From these equations we obtain

$$x = \frac{2a}{a+b} = \frac{2\log 2}{\log 2 + \log 5} = \frac{2\log 2}{\log 10} = 2\log 2 = .60206,$$

$$y = -\frac{ax}{b} = -2\log 5 = -1.39794.$$

13. Put $\log 2 = a, \log 3 = b;$ then
 $ax = by, a(y+1) = b(x-1).$

From these equations we obtain

$$x = \frac{b}{b-a} = \frac{\log 3}{\log 3 - \log 2} = 2.71,$$

$$y = \frac{a}{b-a} = \frac{\log 2}{\log 3 - \log 2} = x-1 = 1.71.$$

14. We have $2\log 2 + \log 7 = a, \log 3 + \log 7 = b, 2-2\log 2 = c.$
 $\therefore 2+\log 7 = a+c,$ so that $\log 3 = b-a-c+2.$
 $\therefore \log 27 = 3(b-a-c+2).$

Again $\log 224 = \log(2^5 \times 7) = 5\log 2 + \log 7.$

$$= \frac{5}{2}(2-c) + (a+c-2) = \frac{1}{2}(2a-3c+6).$$

15. We have $\log(2 \times 11^2) = a, \log(2^3 \times 10) = b, \log(3^2 \times 5) = c.$
 $\therefore 2\log 11 + \log 2 = a, 3\log 2 + 1 = b, 2\log 3 + 1 - \log 2 = c.$

From the last two equations

$$b+c-2 = 2\log 3 + 2\log 2 = \log 36.$$

$$\text{Again, } \log 66 = \log 6 + \log 11 = \frac{1}{2}(b+c-2) + \frac{a-\log 2}{2}$$

$$= \frac{1}{2} \left[b+c-2 + a - \frac{1}{3}(b-1) \right] = \frac{1}{6}(3a+2b+3c-5).$$

MISCELLANEOUS EXAMPLES. E. PAGE 150.

1. First side = $\frac{1}{2}(\cos 60^\circ + \cos 2A - \cos 120^\circ - \cos 2A)$
 $= \frac{1}{2}(\cos 60^\circ - \cos 120^\circ) = \frac{1}{2}.$

2. See Ex. 1, p. 118, and Ex. 3, p. 119.

$$3. b^2 = a^2 + c^2 - 2ac \cos B = 4 + 2 - 2 \cdot 2 \cdot \sqrt{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = 4 - 2\sqrt{3} = (\sqrt{3}-1)^2.$$

$$\therefore b = \sqrt{3}-1.$$

$$\sin A = \frac{a}{b} \sin B = \frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$\therefore A = 45^\circ$, or 135° ; but a is the greatest side, so that $A = 135^\circ$, and $C = 30^\circ$.

4. Each side easily reduces to 1.

$$5. \text{ Here } \frac{a}{b} = \frac{\cos A}{\cos B}; \text{ but in any triangle } \frac{a}{b} = \frac{\sin A}{\sin B}.$$

$$\therefore \frac{\sin A}{\sin B} = \frac{\cos A}{\cos B}, \text{ or } \sin(A-B) = 0;$$

$\therefore A = B$, and the triangle is isosceles.

$$6. (1) \text{ First side} = \frac{1}{2}(\cos 2\theta - \cos 4\theta + \cos 4\theta - \cos 6\theta + \cos 6\theta - \cos 8\theta + \cos 8\theta - \cos 10\theta)$$

$$= \frac{1}{2}(\cos 2\theta - \cos 10\theta) = \sin 6\theta \sin 4\theta.$$

$$(2) \text{ First side} = \frac{2 \sin 2\alpha \cos \alpha + 2 \sin 6\alpha \cos \alpha}{2 \cos 2\alpha \cos \alpha + 2 \cos 6\alpha \cos \alpha}$$

$$= \frac{\sin 2\alpha + \sin 6\alpha}{\cos 2\alpha + \cos 6\alpha} = \frac{2 \sin 4\alpha \cos 2\alpha}{2 \cos 4\alpha \cos 2\alpha} = \tan 4\alpha.$$

$$7. \text{ First side} = \frac{\cos 3\alpha \cos \alpha + \sin 3\alpha \sin \alpha}{\sin \alpha \cos \alpha} = \frac{2 \cos(3\alpha - \alpha)}{2 \sin \alpha \cos \alpha}$$

$$= \frac{2 \cos 2\alpha}{\sin 2\alpha} = 2 \cot 2\alpha.$$

$$8. c^2 = a^2 + b^2 - 2ab \cos C = a^2 + (4 - 2\sqrt{3}) a^2 - a^2 (\sqrt{3}-1) \sqrt{3} = (2-\sqrt{3}) a^2.$$

$$\therefore a^2 = (2+\sqrt{3}) c^2, \text{ and } \sin^2 A = (2+\sqrt{3}) \sin^2 C;$$

$$\therefore \sin^2 A = \frac{2+\sqrt{3}}{4} = \frac{4+2\sqrt{3}}{8};$$

$$\therefore \sin A = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Hence $A = 75^\circ$, or 105° , and the latter value must be taken, as $A = 75^\circ$ would make the triangle isosceles. Hence also $B = 45^\circ$.

$$9. \tan 4\alpha = \frac{2 \tan 2\alpha}{1 - \tan^2 2\alpha}; \text{ substitute } \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

$$\begin{aligned}10. \quad (1) \quad \text{First side} &= a^2(1 - 2\sin^2 B) + b^2(1 - 2\sin^2 A) \\&= a^2 + b^2 - 2a^2\sin^2 B - 2b^2\sin^2 A = a^2 + b^2 - 4a^2\sin^2 B \\&= a^2 + b^2 - 4a\sin B \cdot b \sin A = a^2 + b^2 - 4ab \sin A \sin B.\end{aligned}$$

$$\begin{aligned}(2) \quad \text{First side} &= 2bc(1 + \cos A) + \dots + \dots \\&= (2bc + b^2 + c^2 - a^2) + \dots + \dots = (a + b + c)^2.\end{aligned}$$

11. From the equation $c^4 - 2c^2(a^2 + b^2) + (a^4 + b^4) = 0$ we have

$$\{c^2 - (a^2 + ab\sqrt{2} + b^2)\}\{c^2 - (a^2 - ab\sqrt{2} + b^2)\} = 0.$$

Equating the two factors separately to zero, we get

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}};$$

whence

$$C = 45^\circ \text{ or } 135^\circ.$$

$$12. \quad \text{We have } 2 \cos \frac{3(A+B)}{2} \cos \frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore 2 \cos \left(270^\circ - \frac{3C}{2}\right) \cos \frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore -2 \sin \frac{3C}{2} \cos \frac{3(A-B)}{2} + 1 - 2 \sin^2 \frac{3C}{2} = 1;$$

$$\text{or } -2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} + \sin \frac{3C}{2} \right] = 0;$$

$$2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} - \cos \frac{3(A+B)}{2} \right] = 0;$$

$$4 \sin \frac{3A}{2} \sin \frac{3B}{2} \sin \frac{3C}{2} = 0.$$

Since A, B, C are the angles of a triangle we must have one of the angles $\frac{3A}{2}, \frac{3B}{2}$, or $\frac{3C}{2}$ equal to 180° . That is, one of the angles of the triangle must be 120° .

EXAMPLES. XV. a. PAGE 155.

1. $\log 49517$	=	4.6947543	2.	$\log 3.4714$	=	$.5405047$
$\log 49516$	=	4.6947456		$\log 3.4713$	=	$.5404921$
diff. for 1	=	$\frac{87}{.34}$		diff. for .0001	=	126
		$\frac{348}{261}$				$.026$
diff. for .34 =		$\frac{29}{58}$				$\frac{756}{252}$
$\log 49516$	=	4.694756		diff. for .0000026 =		$\frac{3276}{}$
$\log 49516.34 =$	4.694786			$\log 3.4713$	=	$.5404921$
$\therefore \log 4951634 =$	6.694786			$\log 3.4713026 =$	$.5404924$	

$$\begin{array}{rcl} 3. \quad \log 28497 & = & 4.4547991 \\ \log 28496 & = & 4.4547839 \\ \text{diff. for } 1 & = & \underline{\begin{array}{r} 152 \\ \cdot 14 \\ \hline 608 \\ 152 \\ \hline 2128 \end{array}} \\ \text{diff. for } \cdot 14 & = & \underline{2128} \\ \log 28496 & = & 4.4547839 \\ \log 28496 \cdot 14 & = & 4.4547860 \\ \therefore \log 2849614 & = & 6.4547860. \end{array}$$

$$\begin{array}{rcl} 5. \quad \log 60814 & = & 4.7840036 \\ & & \underline{\begin{array}{r} 6 \\ 5 \\ \hline 432 \\ 360 \end{array}} \\ \therefore \log 6081465 & = & \underline{6.7840083} \end{array}$$

$$\begin{array}{rcl} 7. \quad \log x & = & 2.8283676 \\ \log 67354 & = & 2.8283634 \\ \text{diff.} & = & \underline{42}, \\ \text{and diff. for } 1 & = & 64; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{42}{64} = \frac{21}{32} = .66; \\ \therefore x & = & 673.5466. \end{array}$$

$$\begin{array}{rcl} 9. \quad \log x & = & 3.9184377 \\ \log .0082877 & = & 3.9184340 \\ \text{diff.} & = & \underline{37}, \\ \text{and diff. for } 1 & = & 52; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{37}{52} = .71; \\ \therefore x & = & .008287771. \end{array}$$

$$\begin{array}{rcl} 11. \quad \log x & = & \frac{1}{7} \log 142.71 \\ & = & \frac{1}{7} (2.1544544) \\ & = & .3077792 \\ \log 2.0313 & = & .3077741 \\ \text{diff.} & = & \underline{51}, \\ \text{and diff. for } 1 & = & 213; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{51}{213} = .24; \\ \therefore x & = & 2.031324. \end{array}$$

$$\begin{array}{rcl} 4. \quad \log 57.634 & = & 1.7606788 \\ \log 57.633 & = & 1.7606712 \\ \text{diff. for } .001 & = & \underline{\begin{array}{r} 76 \\ \cdot 25 \\ \hline 380 \\ 152 \\ \hline 1900 \end{array}} \\ \text{diff. for } .00025 & = & \underline{1900} \\ \log 57.633 & = & 1.7606712 \\ \log 57.63325 & = & 1.7606731 \end{array}$$

$$\begin{array}{rcl} 6. \quad \log x & = & 4.7461735 \\ \log 55740 & = & 4.7461670 \\ \text{diff.} & = & \underline{65}, \\ \text{and diff. for } 1 & = & 78; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{65}{78} = \frac{5}{6} = .83; \\ \therefore x & = & 55740.83. \end{array}$$

$$\begin{array}{rcl} 8. \quad \log x & = & 2.0288435 \\ \log .010686 & = & 2.0288152 \\ \text{diff.} & = & \underline{283}, \\ \text{and diff. for } 1 & = & 406; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{283}{406} = .7; \\ \therefore x & = & .0106867. \end{array}$$

$$\begin{array}{rcl} 10. \quad \log x & = & 1.4034508 \\ \log .25319 & = & 1.4034465 \\ \text{diff.} & = & \underline{43}, \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{43}{172} = \frac{1}{4} = .25; \\ \therefore x & = & .2531925. \end{array}$$

$$\begin{array}{rcl} 12. \quad \log 13.894 & = & 1.1428273 \\ & & \underline{\begin{array}{r} 9 \\ 2 \\ \hline 281 \\ 626 \\ \hline 8 \end{array}} \underline{1.1428561} \\ \log x & = & .1428570 \\ \log 1.3894 & = & .1428273 \\ \text{diff.} & = & \underline{297}, \\ \text{and diff. for } 1 & = & 313; \\ \therefore \text{prop}^l. \text{ increase} & = & \frac{297}{313} = .95; \\ \therefore x & = & 1.389495. \end{array}$$

$$\begin{array}{ll}
 13. \log 24244 = 5.3846043 & 14. \log 20691 = 6.3157815 \\
 & 7 \quad 125|3 \\
 & \quad 14 \overline{)5.3846168} \\
 \log x & = .3846155 \\
 \log 2.4244 & = .3846043 \\
 \text{diff.} & = \underline{112}, \\
 \therefore \text{prop}^l. \text{increase} & = \frac{112}{179} = .63; \\
 \therefore x & = 2.424463. \\
 & \\
 & 14. \log 20691 = 6.3157815 \\
 & 3 \quad 63|0 \\
 & \quad 8 \quad 16|80 \\
 \log x & = .3157895 \\
 \log 2.0691 & = .3157815 \\
 \text{diff.} & = \underline{80}, \\
 \therefore \text{prop}^l. \text{increase} & = \frac{80}{210} = \frac{8}{21} = .38; \\
 \therefore x & = 2.069138.
 \end{array}$$

EXAMPLES. XV. b. PAGE 159.

1. $\sin 38^\circ 3' = .6163489$; diff. for $60'' = 2291$.

$$\text{prop}^l. \text{increase} = \frac{35}{60} \times 2291 = \underline{.6164825}$$

3. $\text{cosec } 55^\circ 21' = 1.2155978$; diff. for $60'' = 2443$.

$$\text{prop}^l. \text{decrease} = \frac{28}{60} \times 2443 = \underline{1.2154838}$$

4. $\sec \theta - \sec 62^\circ 42' = 6321$;

diff. for $60'' = 12296$;

and $\frac{6321}{12296} \times 60'' = 31''$;

$\therefore \theta = 62^\circ 42' 31''$.

5. $\cos 30^\circ 40' - \cos \theta = 560$;

diff. for $60'' = 1484$;

and $\frac{560}{1484} \times 60'' = 23''$;

$\therefore \theta = 30^\circ 40' 23''$.

6. $\cot 48^\circ 45' - \cot \theta = 3762$;

diff. for $60'' = 5145$;

and $\frac{3762}{5145} \times 60'' = 44''$;

$\therefore \theta = 48^\circ 45' 44''$.

7. $L \sin 44^\circ 17' = 9.8439842$; diff. for $60'' = 1295$.

$$\text{prop}^l. \text{increase} = \frac{33}{60} \times 1295 = \underline{9.8440554}$$

9. $L \cos 55^\circ 30' = 9.7531280$; diff. for $60'' = 1838$.

$$\text{prop}^l. \text{decrease} = \frac{24}{60} \times 1838 = \underline{9.7530545}$$

10. $L \sin \theta - L \sin 44^\circ 17' = 176$; 11. $L \cos 55^\circ 30' - L \cos \theta = 1205$;
 diff. for $60''$ $= 1295$; diff. for $60''$ $= 1838$;
 and $\frac{176}{1295} \times 60'' = 8''$; and $\frac{1205}{1838} \times 60'' = 39''$;
 $\therefore \theta = 44^\circ 17' 8''$. $\therefore \theta = 55^\circ 30' 39''$.

12. $L \tan 24^\circ 50' = 9.6653662$; diff. for $60'' = 3313$.

$$\text{prop}^l. \text{ increase} = \frac{52.5}{60} \times 3313 = \frac{2899}{9.6653661}$$

13. The required angle is $42.5''$ less than $40^\circ 5'$;

$$\therefore \text{prop}^l. \text{ increase} = \frac{42.5}{60} \times 1502 = 1064$$

$$L \operatorname{cosec} 40^\circ 5' = 10.1911808$$

$$L \operatorname{cosec} 40^\circ 4' 17.5'' = \underline{\underline{10.1912872}}$$

EXAMPLES. XV. c. PAGE 161.

1. $\log 300.26 = 2.4774975$

1	15
8	11
9	6
4	50
	22
	·3746609
2	·3746567
	42
	37

$\log .0078915 = \underline{\underline{3.8971596}}$

$\log 2.3695 = \underline{\underline{·3746567}}$

Thus the product = 2.36952.

2. $\log 235.67 = 2.3723043$

8	148
3	56
8	36
	98
	4.9260131
	4.9260130

$\log 357.84 = 2.5536889$

$\log 84336 = \underline{\underline{4.9260130}}$

Thus the product is 84336.

3.

$$\begin{array}{r}
 \log 153.24 = 2.1853721 \\
 1 \quad 28 \\
 9 \quad 256 \\
 \log 2.8632 = .4568517 \\
 5 \quad 76 \\
 0 \quad 0 \\
 3 \quad 46 \\
 \log .075836 = 2.8798754 \\
 4 \quad 23 \\
 6 \quad 34 \\
 \hline
 1.5221148 \\
 \log 33.274 = 1.5221050 \\
 98 \\
 7 \quad 91 \\
 \hline
 5 \quad 70 \\
 \hline
 5 \quad 65
 \end{array}$$

Thus the product is 33.27475.

$$\begin{array}{r}
 4. \quad \log 1.0304 = .0130059 \\
 0 \quad 0 \\
 5 \quad 211 \\
 1 \quad 42 \\
 \hline
 .0130081 \\
 \text{subtract} \quad 1.4328656 \\
 \hline
 2.5801425 \\
 \log .038031 = 2.5801377 \\
 48 \\
 4 \quad 46 \\
 \hline
 20 \\
 2 \quad 23
 \end{array}$$

$$\begin{array}{r}
 \log 27.093 = 1.4328571 \\
 5 \quad 81 \\
 2 \quad 32 \\
 4 \quad 64 \\
 \hline
 1.4328656
 \end{array}$$

Thus the quotient is .03803142.

$$\begin{array}{r}
 5. \quad \log 357.83 = 2.5536767 \\
 6 \quad 73 \\
 4 \quad 48 \\
 \hline
 2.5536845 \\
 \bar{3.5037539} \\
 \overline{5.0499306} \\
 \log 11218 = 4.0499154 \\
 4 \quad 152 \\
 \hline
 155
 \end{array}$$

$$\begin{array}{r}
 \log .0031897 = \overline{3.5037498} \\
 3 \quad 41 \\
 \hline
 \overline{3.5037539}
 \end{array}$$

Thus the quotient is 112184.

$$\begin{array}{rcl}
 6. \quad \log 21.856 & = 1.3395707 & \log .017834 = \bar{2}.2512488 \\
 & \begin{array}{r} 3 \\ 2 \\ \hline 1 \end{array} & \begin{array}{r} 5 \\ \hline 122 \\ \hline \bar{2}.2512610 \end{array} \\
 & \begin{array}{r} 60 \\ 40 \\ \hline 1.3395771 \end{array} & \\
 \text{subtract} & \bar{2}.2512610 & \\
 \log x & = 3.0883161 & \\
 \log 1225.5 & = 3.0883133 & \\
 & \begin{array}{r} 0 \\ 8 \\ \hline 280 \\ 284 \end{array} & \\
 \therefore x = 1225.508. & &
 \end{array}$$

$$\begin{array}{rcl}
 7. \quad \log 3.7895 & = .5785819 & \\
 & \begin{array}{r} 6 \\ \hline 69 \end{array} & \\
 \log .053687 & = \bar{2}.7298691 & \\
 & \begin{array}{r} 2 \\ \hline 16 \\ \hline 1.3084595 \end{array} & \\
 \log .0072916 & = \bar{3}.8628228 & \\
 & \begin{array}{r} 14456367 \\ \hline 1.4456353 \end{array} & \\
 \log 27.902 & = 1.4456353 & \\
 & \begin{array}{r} 0 \\ 9 \\ \hline 140 \\ 140 \end{array} &
 \end{array}$$

Thus the required value is 27.90209.

$$\begin{array}{rcl}
 8. \quad \log .83410 & = \bar{1}.9212181 & \\
 & \begin{array}{r} 0 \\ 3 \\ 9 \\ \hline 16 \\ |47 \\ \hline \end{array} & \\
 & \begin{array}{r} 0 \\ 1 \\ |6 \\ 47 \\ \hline \end{array} & \\
 & \begin{array}{r} 1.9212183 \\ 3 \\ \hline \end{array} & \\
 \log .58030 & = \bar{1}.7636549 & \\
 & \begin{array}{r} 3 \\ \hline 23 \\ 22 \end{array} &
 \end{array}$$

Thus the cube is .580303.

$$\begin{array}{rcl}
 9. \quad \log 15063 & = 4.1779115 & \\
 & \begin{array}{r} 0 \\ 1 \\ 8 \\ \hline 29 \\ 230 \\ \hline \end{array} & \\
 & \begin{array}{r} 0 \\ 2 \\ |9 \\ 230 \\ \hline \end{array} & \\
 & \begin{array}{r} 5 | 4.1779120 \\ \hline .8355824 \end{array} & \\
 \log 6.8482 & = .8355764 & \\
 & \begin{array}{r} 9 \\ 3 \\ \hline 60 \\ 58 \\ \hline 20 \\ 19 \end{array} &
 \end{array}$$

Thus the fifth root is 6.848293.

10.	$\log 384\cdot73 = 2\cdot5851561$	$\log 15\cdot732 = 1\cdot1967839$
	1 11 $5 \mid 2\cdot5851572$ ·5170314	4 111 $13 \mid 1\cdot1967950$ ·0920612
	$\log 3\cdot2887 = \underline{\cdot5170243}$	$\log 1\cdot2361 = \underline{\cdot0920536}$
	71 5 66 — 50 4 53	2 — 60 2 — 70

Thus $\sqrt[5]{384\cdot73} = 3\cdot288754$.

Thus $\sqrt[10]{15\cdot7324} = 1\cdot236122$.

11.	$\log 1034\cdot3 = 3\cdot0146465$	$\log 35324 = 5\cdot5480699$
	9 6 3 2 add log 2273·5	6 3 $3 \mid 5\cdot5480773$ 1·8493591
	379 253 126 3·0146871 1·5073435 1·8493591 3·3567026 = 3·3566950 — 76 4 — 76	74 — 1·8493591

Thus the product is 2273·54.

12. Let $a = 1\cdot0356270$ and $b = \cdot7503269$; then $a^2 - b^2 = (a + b)(a - b)$, and $a + b = 1\cdot7859539$, $a - b = \cdot2853001$.

$\log 1\cdot7859$	$= \cdot2518571$
5 3 9 0 1	122 73 219 0 15
$\log \cdot28530$	$= \cdot1\cdot4553018$
2 8	$\overline{1\cdot7071722}$ $\overline{1\cdot7071698}$ 24 17 — 70 69

Thus the difference is ·5095328.

13. $\log x = \frac{3}{5} \log 34.7326 + \frac{1}{6} \log 2.53894 - \frac{1}{5} \log 4.39682.$

$$\begin{array}{rcl} \log 2.5389 & = & .4046456 \\ & & 68 \\ & 4 & \overline{)4046524} \\ & 6 & \overline{)0674421} \\ \text{add} & & .7958146 \\ & & \overline{.8632567} \\ \log 7.2988 & = & \overline{.8632515} \\ & & 52 \\ & 9 & \overline{54} \end{array}$$

Thus $x = 7.29889.$

$$\begin{array}{rcl} \log 4.3968 & = & .6431367 \\ & & 20 \\ & 2 & \overline{)6431387} \\ \log 34.732 & = & 1.5407298 \\ & 6 & \overline{75} \\ & & \overline{1.5407373} \\ & & 3 \\ \text{subtract} & & \overline{4.6222119} \\ & & .6431387 \\ & 5 & \overline{3.9790732} \\ & & \overline{.7958146} \end{array}$$

14. $\log .0037258 = \bar{3}.5712195$

$$\begin{array}{rcl} 1 & & 12 \\ 6 & & 70 \\ 9 & & \overline{1.05} \\ & & \overline{3.5712215} \\ \text{add} & & \overline{1.7505167} \\ & 2 & \overline{)3.3217382} \\ & & \overline{2.6608691} \\ \log .045800 & = & \overline{2.6608655} \\ & & 36 \\ & 3 & \overline{29} \\ & 7 & \overline{70} \\ & & \overline{67} \end{array}$$

$$\begin{array}{rcl} \log .56301 & = & \bar{1}.7505161 \\ 0 & & 0 \\ 7 & & 54 \\ 8 & & \overline{62} \\ & & \overline{1.7505167} \end{array}$$

Thus the mean proportional is .04580037.

15. If x be the required number, we have $x = \frac{.03751786}{(.43607528)^2},$

$$\begin{array}{rcl} \log .037517 & = & \bar{2}.5742281 \\ 8 & & 93 \\ 6 & & 70 \\ & & \overline{\bar{2}.5742381} \\ & & \overline{1.2791230} \\ & & \overline{1.2951151} \\ \log .19729 & = & \overline{1.2951051} \\ & & 100 \\ 4 & & 88 \\ & 5 & \overline{120} \\ & & \overline{110} \end{array}$$

$$\begin{array}{rcl} \log .43607 & = & \bar{1}.6395562 \\ 5 & & 50 \\ 2 & & 20 \\ 8 & & \overline{79} \\ & & \overline{1.6395615} \\ & & 2 \\ & & \overline{1.2791230} \end{array}$$

Thus $x = .1972945.$

16. If x be the required number, we have

$$x = \frac{29.302564 \times 33025107}{56712.43}.$$

$\log 29.302$	$= 1.4668973$	$\log 56712$	$= 4.7536750$
5 6 4	74 89 59	4 3	31 23
$\log .33025$	$= 1.5188428$		$\underline{4.7536783}$
1 0 7	13 0 92		
	$\underline{\underline{.9857498}}$		
subtract	$\underline{4.7536783}$		
	$\underline{\underline{4.2320715}}$		
$\log .00017063$	$= 4.2320554$		
6	161 152 90 76		
	$\underline{\underline{}}$		

Thus the fourth proportional is .0001706363.

17. Let x be the required number, then

$$x = \sqrt{(.035689)^{\frac{2}{5}} \times (2.879432)^{\frac{3}{7}}}.$$

$\log 2.8794$	$= -4593020$	$\log .035689 = \frac{1}{5} (2.5525344)$
3 2	45 30 $\underline{-4593068}$	$= \bar{1.7105069}$
	3	
	14 [$\underline{1.3779204}$	
	$\underline{.0984229}$	
	$\bar{1.7105069}$	
	$\underline{1.8089298}$	
$\log .64406$	$= \underline{1.8089263}$	
5	35 $\underline{34}$	

Thus the geometric mean is .644065.

18. Here $x = \frac{(7836.43)^{\frac{1}{4}} \times (357.814)^{\frac{1}{3}}}{(32.7812)^{\frac{1}{2}}}.$

$$\begin{array}{rcl} \log 7836.4 & = 3.8941166 \\ 3 & & 17 \\ 4 | 3.8941183 & & \\ & -9735295 & \\ & \hline & .5107315 \\ \text{add} & & \\ & 1.4842610 & \\ \text{subtract} & -5052083 & \\ & \hline & .9790527 \\ \log 9.5291 & = .9790519 & \\ & \hline & 8 \\ 2 & & 9 \end{array}$$

$$\begin{array}{rcl} \log 32.781 & = 1.5156222 \\ 2 & & 26 \\ 3 | 1.5156248 & & \\ & -5052083 & \\ & \hline & \\ \log 357.81 & = 2.5536525 \\ 4 & & 49 \\ 5 | 2.5536574 & & \\ & -5107315 & \\ & \hline & \end{array}$$

Thus the fourth proportional is 9.52912.

19. $\log \sin 27^\circ 13' = \bar{1}.6602550$

$$\begin{array}{rcl} \frac{12}{60} \times 2455 & = & 491 \\ & & \hline & & \\ & & \bar{1}.6603041 \\ & & \bar{1}.8414768 \\ & & \hline & & \\ & & \bar{1}.5017809 \\ \log .31752 & = & \bar{1}.5017711 \\ & & \hline & & \\ & & 98 \\ 7 & & 96 \\ & & \hline & & \\ & & 20 \\ 1 & & 14 \end{array}$$

$$\begin{array}{rcl} \log \cos 46^\circ 2' & = \bar{1}.8415095 \\ \text{subtract } \frac{15}{60} \times 1310 & = & 327 \\ & & \hline & & \\ & & \bar{1}.8414768 \end{array}$$

Thus the required value is .3175271.

20. $\cot 97^\circ 14' 16'' = -\cot 82^\circ 45' 44'',$
 $\sec 112^\circ 13' 5'' = -\sec 67^\circ 46' 55''.$

$$\begin{array}{rcl} \log \sec 67^\circ 46' & = .4220725 \\ \frac{11}{12} \times 3092 & = & 2834 \\ & & \hline & & \\ & & \bar{4}223559 \\ & & \hline & & \\ & & \bar{1}.1038011 \\ & & \hline & & \\ & & \bar{1}.5261570 \\ \log .33585 & = & \bar{1}.5261454 \\ & & \hline & & \\ & & 116 \\ 9 & & \underline{116} \end{array}$$

$$\begin{array}{rcl} \log \cot 82^\circ 45' & = \bar{1}.1045420 \\ \frac{11}{15} \times 10103 & = & 7409 \\ & & \hline & & \\ & & \bar{1}.1038011 \end{array}$$

Thus the required value is .335859.

21.	$\log \sin 20^\circ 13' = \bar{1} \cdot 5385375$	$\log \cot 47^\circ 53' = \bar{1} \cdot 9562154$
	$\frac{20}{60} \times 3429 = 1143$	$\text{subtract } \frac{15}{60} \times 2540 = 635$
	$\log \sec 42^\circ 15' = \bar{1} \cdot 1306403$	$\underline{\underline{1 \cdot 9561519}}$
	$\frac{30}{60} \times 1148 = 574$	$\text{add } \underline{\underline{1 \cdot 6693495}}$
	$1 \cdot 6693495$	$\underline{\underline{1 \cdot 6255014}}$
		$\log .42218 = \underline{\underline{1 \cdot 6254977}}$
		$\frac{37}{3}$
		$\frac{31}{60}$
		$\frac{6}{62}$

Thus the required value is .4221836.

22.	$\log 324 \cdot 13 = 2 \cdot 5107192$	$\log \sin 113^\circ 14' 16''$
	$\frac{6}{8} \quad \frac{80}{10} \quad 7$	$= \log \sin 66^\circ 45' 44'',$
	$\log 417 \cdot 24 = 2 \cdot 6203859$	$\log \sin 66^\circ 45' = \bar{1} \cdot 9632168$
	$\frac{3}{1} \quad \frac{31}{1} \quad 0$	$\frac{44}{60} \times 543 = \underline{\underline{398}}$
	$\underline{\underline{5 \cdot 1311174}}$	$\underline{\underline{1 \cdot 9632566}}$
	$\underline{\underline{1 \cdot 9632566}}$	$\underline{\underline{5 \cdot 0943740}}$
	$\underline{\underline{5 \cdot 0943740}}$	$\log 12427 = 5 \cdot 0943663$
		$\frac{77}{2} \quad \frac{70}{2} \quad \frac{70}{2}$

Thus the required value is 12427.2.

23. Here $a = \frac{b \sin A}{\sin B}$, and $\sin B = \sin 60^\circ 45' 42''$.

log sin 35° 15'	= 1·7612851	log sin 60° 45' = 1·9407634
$\frac{33}{60} \times 1787 = 983$		$\frac{42}{60} \times 708 = \underline{\underline{496}}$
	$\underline{\underline{1 \cdot 7613834}}$	$\underline{\underline{1 \cdot 9408130}}$
	$\underline{\underline{1 \cdot 9408130}}$	
	$\underline{\underline{1 \cdot 8205704}}$	
log 378·25	= 2·5777789	
	$\underline{\underline{2 \cdot 3983493}}$	
log 250·23	= 2·3983394	
	$\underline{\underline{99}}$	
5	$\underline{\underline{87}}$	
	$\underline{\underline{120}}$	
7	$\underline{\underline{121}}$	

Thus $a = 250 \cdot 2357$.

$$\begin{array}{ll}
 24. \quad (1) \log \tan \theta = \frac{1}{3} (\log 5 - \log 12) & (2) \quad (3 \sin \theta - 1)(\sin \theta + 1) = 0. \\
 \log 5 = & \therefore \sin \theta = \frac{1}{3} \text{ or } -1. \\
 \log 12 = & \log \sin \theta = -\log 3 \\
 & = 1.5228787 \\
 3 | \overline{1.6197888} & \log \sin 19^\circ 28' = \frac{1.5227811}{976} \\
 \log \tan \theta = & 976 \\
 \log \tan 36^\circ 45' = & 3572 \times 60'' = 16''. \\
 & \therefore \theta = 19^\circ 28' 16''. \\
 \underline{961} & \\
 \underline{2634} \times 60'' = 22''. & \\
 \therefore \theta = 36^\circ 45' 22''. &
 \end{array}$$

$$25. \quad x = \sin 23^\circ 18' 5'' \times \cot 38^\circ 15' 13'' \times \cos 28^\circ 17' 25''.$$

$$\begin{array}{ll}
 \log \cot 38^\circ 15' = -1.032884 & \log \cos 28^\circ 17' = 1.9447862 \\
 \text{subtract } \frac{13}{60} \times 2598 = \frac{562}{\cdot 1032322} & \text{subtract } \frac{25}{60} \times 680 = \frac{283}{1.9447579} \\
 & \\
 \log \sin 23^\circ 18' = 1.5971965 & \\
 \frac{5}{60} \times 2932 = \frac{244}{1.6452110} & \\
 \log \cdot 44178 = \frac{1.6452061}{49} & \\
 & \underline{5} \quad \underline{49}
 \end{array}$$

Thus $x = \cdot 441785$.

$$\begin{array}{ll}
 26. \quad \log \cos 32^\circ 47' = & 1.9246535 \\
 & \underline{2} \\
 & \overline{1.8493070} \\
 \log \cot 41^\circ 19' = & \cdot 0559928 \\
 & \\
 & 3 | \overline{1.9052998} \\
 & \overline{1.9684333} \\
 \log \sin 68^\circ 25' = & \frac{1.9684286}{47} \\
 & \\
 & \frac{47}{499} \times 60'' = 6''.
 \end{array}$$

Thus $\theta = 68^\circ 25' 6''$.

EXAMPLES. XV. d. PAGE 163 C.

1. $\log 2834 = 3\cdot4524$
 $\log 17\cdot62 = 1\cdot2460$
 $\log x = \underline{4\cdot6984}$;
whence $x = 49940.$
2. $\log 8\cdot034 = \underline{0\cdot9049}$
 $\log 1893 = \underline{3\cdot2772}$
 $\log x = \underline{4\cdot1821};$
whence $x = 15210.$
3. $\log .00567 = \underline{\overline{3\cdot7536}}$
 $\log .0297 = \underline{\overline{2\cdot4728}}$
 $\log x = \underline{\overline{4\cdot2264}};$
whence $x = .0001685.$
4. $\log 3\cdot7 = \underline{.5682}$
 $\log 8\cdot9 = \underline{.9494}$
 $\log .023 = \underline{\overline{2\cdot3617}}$
 $\log x = \underline{\overline{1\cdot8793}};$
whence $x = .7573.$
5. $\log 31\cdot9 = 1\cdot5038$
 $\log 1\cdot51 = \underline{.1790}$
 $\log 9\cdot7 = \underline{.9868}$
 $\log x = \underline{\overline{2\cdot6696}};$
whence $x = 467\cdot3.$
6. $\log 43 = 1\cdot6335$
 $\log 8\cdot07 = \underline{.9069}$
 $\log .0392 = \underline{\overline{2\cdot5933}}$
 $\log x = \underline{\overline{1\cdot1337}};$
whence $x = 13\cdot60.$
7. $\log 17\cdot3 = 1\cdot2380$
 $\log 294\cdot8 = \underline{\overline{2\cdot4695}}$
 $\log x = \underline{\overline{2\cdot7685}};$
whence $x = .05868.$
8. $\log 2\cdot035 = \underline{.3086}$
 $\log 837\cdot6 = \underline{\overline{2\cdot9230}}$
 $\log x = \underline{\overline{3\cdot3856}};$
whence $x = .00243.$
9. $\log .2179 = \underline{\overline{1\cdot3383}}$
 $\log .08973 = \underline{\overline{2\cdot9529}}$
 $\log x = \underline{.3854};$
whence $x = 2\cdot429.$
10. $\log 487 = 2\cdot6875$
 $\log 6398 = 3\cdot8060$
 $\log x = \underline{\overline{2\cdot8815}};$
whence $x = .07612.$
11. $\log 2\cdot38 = \underline{.3766}$
 $\log 3\cdot901 = \underline{\overline{.5912}}$
 $\log x = \underline{\overline{.9678}}$
 $\log 4\cdot83 = \underline{\overline{.6839}}$
 $\log x = \underline{\overline{.2839}};$
whence $x = 1\cdot923.$
12. $\log 14\cdot72 = 1\cdot1679$
 $\log 38\cdot05 = \underline{\overline{1\cdot5804}}$
 $\log x = \underline{\overline{2\cdot7483}}$
 $\log 387\cdot9 = \underline{\overline{2\cdot5887}}$
 $\log x = \underline{\overline{.1596}};$
whence $x = 1\cdot444.$
13. $\log 925\cdot9 = 2\cdot9665$
 $\log 1\cdot597 = \underline{\overline{.2034}}$
 $\log x = \underline{\overline{3\cdot1699}}$
 $\log 74\cdot03 = \underline{\overline{1\cdot8694}}$
 $\log x = \underline{\overline{1\cdot3005}}; \text{ whence } x = 19\cdot97.$
14. $\log 15\cdot38 = 1\cdot1869$
 $\log .0137 = \underline{\overline{2\cdot1367}}$
 $\log x = \underline{\overline{1\cdot3236}}$
 $\log x = \underline{\overline{.0207}}$
 $\log x = \underline{\overline{1\cdot3029}}; \text{ whence } x = .2008.$
- $\log 276 = 2\cdot4409$
 $\log .0038 = \underline{\overline{3\cdot5798}}$
 $\log x = \underline{\overline{.0207}}$

15. $\log 2.31 = \underline{.3636}$ $\log .0561 = \bar{2}.7490$
 $\log .037 = \bar{2}.5682$ $\log 3.87 = .5877$
 $\log 1.43 = \underline{\underline{.1553}}$ $\log .0091 = \bar{3}.9590$
 $\quad \quad \quad \bar{1}.0871$
 $\quad \quad \quad \bar{3}.2957$

$\log x = \bar{1}.7914$; whence $x = 61.86$.

16. $\log x = \frac{1}{2} \log 5.1 = \frac{1}{2} (.7076) = .3538$; whence $x = 2.258$.

17. $\log x = \frac{1}{3} \log 11 = \frac{1}{3} (1.0414) = .3471$; whence $x = 2.224$.

18. $\log x = \frac{1}{3} \log 82.56 = \frac{1}{3} (1.9168) = .6839$; whence $x = 4.354$.

19. $\log x = \frac{1}{4} \log 10.15 = \frac{1}{4} (1.0064) = .2516$; whence $x = 1.784$.

20. $\log x = 4 \log .097 = 4 (\bar{2}.9868) = \bar{5}.9472$; whence $x = .00008855$.

21. $5 \log 2.301 = .3619 \times 5 = 1.8095$; whence $x = 64.49$.

22. $\frac{2}{3} \log 51.32 = \frac{1.7103 \times 2}{3} = 1.1402$; whence $x = 13.81$.

23. $\frac{4}{7} \log .089 = \frac{\bar{2}.9494 \times 4}{7} = \bar{1}.3997$; whence $x = .2510$.

24. $\log .0137 = \bar{2}.1367$
 $\log .0296 = \bar{2}.4713$
 $\quad \quad \quad = \bar{4}.6080$
 $\log 873.5 = \underline{\underline{2}.9412}$
 $\quad \quad \quad \bar{2}) \bar{7}.6668$
 $\quad \quad \quad \bar{4}.8334$; whence $x = .0006814$.

25. $\log 83 = 1.9191$ $\log 127 = 2.1038$
 $\frac{1}{3} \log 92 = \underline{\underline{.6546}}$ $\frac{1}{5} \log 246 = \underline{\underline{.4782}}$
 $\quad \quad \quad \bar{2}.5737$
 $\quad \quad \quad \bar{2}.5820$
 $\log x = \bar{1}.9917$; whence $x = .9811$.

26. $\log .678 = \bar{1}.8312$
 $\log 9.01 = \underline{\underline{.9547}}$
 $\quad \quad \quad \bar{7}.8559$
 $\log .0234 = \bar{2}.3692$
 $\quad \quad \quad \bar{2}) \bar{2}.4167$
 $\log x = 1.2084 = \log 16.15$;
 $\therefore x = 16$, to nearest integer.

27. (i) If x is the mean proportional between 2.87 and 30.08,

$$\begin{aligned}x &= \sqrt{2.87 \times 30.08} \\ \log 2.87 &= .4579 \\ \log 30.08 &= 1.4782 \\ 2) \overline{1.9361} \\ \log x &= .9680; \text{ whence } x = 9.29.\end{aligned}$$

- (ii) If x is the third proportional to 0.0238 and 7.805,

$$\begin{aligned}x \times 0.0238 &= (7.805)^2; \therefore x = \frac{(7.805)^2}{0.0238}. \\ 2 \log 7.805 &= 1.7848 \\ \log 0.0238 &= \bar{2}.3766 \\ \log x &= \bar{3}.4082; \text{ whence } x = 2560.\end{aligned}$$

28. Here $x = \sqrt[3]{347.3} \times \sqrt[5]{256.4}^{\frac{1}{2}}$
 $= (347.3)^{\frac{1}{3}} \times (256.4)^{\frac{1}{10}}.$
- $$\begin{aligned}\frac{1}{6} \log 347.3 &= .4234 \\ \frac{1}{10} \log 256.4 &= \underline{.2409} \\ \log x &= \underline{.6643}; \text{ whence } x = 4.616.\end{aligned}$$

29. $\begin{aligned}5 \log x + 3 \log y &= \log 5, \\ 2 \log x + 7 \log y &= \log 11.\end{aligned}$

These equations give

$$\begin{aligned}\log x &= \frac{7 \log 5 - 3 \log 11}{29}, \quad \log y = \frac{5 \log 11 - 2 \log 5}{29}. \\ 7 \log 5 &= 4.8930 & 5 \log 11 &= 5.2070 \\ 3 \log 11 &= 3.1242 & 2 \log 5 &= 1.3980 \\ 29) \overline{1.7688} & (\cdot06099 & 29) \overline{3.8090} & (\cdot1313 \\ 288 & & 90 & \\ 27 & & 39 & \\ & & 100 & \\ \therefore \log x &= \cdot0610; & \therefore \log y &= \cdot1313; \\ \text{whence } x &= 1.151. & \text{whence } y &= 1.353.\end{aligned}$$

30. $\begin{aligned}\log l &= \log 2.863 = .4569 \\ \log g &= \log 32.19 = 1.5077 \\ 2) \overline{2.9492} \\ \log \sqrt{\frac{l}{g}} &= \bar{1}.4746 \\ \log 2 &= .3010 \\ \log \pi &= \underline{.4972} \\ & \quad \cdot2728; \\ \text{whence the required value} &= 1.874.\end{aligned}$

31.

$$\begin{aligned}\log m &= \log 18.34 = 1.2634 \\ \log v^2 &= 2 \log 35.28 = \overline{3.0950} \\ &\quad \overline{4.3584} \\ \log 2 &= \overline{\cdot 3010} \\ \log \frac{1}{2} mv^2 &= 4.0574; \text{ whence } \frac{1}{2} mv^2 = 11410.\end{aligned}$$

32. (i)

$$\begin{aligned}\log p &= \log 93.75 = 1.9719 \\ \log r^n &= 4 \log 1.03 = \overline{\cdot 0512} \\ \log pr^n &= \overline{2.0231}; \text{ whence } pr^n = 105.4.\end{aligned}$$

(ii)

$$\begin{aligned}\log r^3 &= 3 \log 5.875 = 2.3070 & \log 355 &= 2.5502 \\ \log \pi &= \overline{\cdot 4971} & \log 113 &= 2.0531 \\ \log 4 &= \overline{\cdot 6021} & \log \pi &= \overline{\cdot 4971} \\ &\quad \overline{3.4062} \\ \log 3 &= \overline{\cdot 4771} \\ \log \frac{4}{3} \pi r^3 &= 2.9291; \text{ whence } \frac{4}{3} \pi r^3 = 849.4.\end{aligned}$$

33.

$$\begin{aligned}\log m &= \log 33.47 = 1.5246 & \log g &= \log 32.19 = 1.5077 \\ \log v^2 &= \log 3600 = \overline{3.5563} & \log r &= \log 9.6 = \overline{\cdot 9823} \\ &\quad \overline{5.0809} & &\quad \overline{2.4900} \\ &\quad \overline{2.4900} \\ \log F &= \overline{2.5909}; \text{ whence } F = 389.8.\end{aligned}$$

34.

$$r^3 = \frac{3 \times 537.6}{4 \times 3.1416}.$$

$$\begin{aligned}\log 3 &= \overline{\cdot 4771} & \log 4 &= \overline{\cdot 6021} \\ \log 537.6 &= 2.7305 & \log 3.1416 &= \overline{\cdot 4971} \\ &\quad \overline{3.2076} & &\quad \overline{1.0992} \\ &\quad \overline{1.0992} \\ 3) \overline{2.1084} \\ \log r &= \overline{.7028}; \text{ whence } r = 5.044.\end{aligned}$$

35.

$$f = \frac{2s}{t^2} = \frac{578.6 \times 8^2}{31^2}.$$

$$\begin{aligned}\log 578.6 &= 2.7624 \\ \log 64 &= \overline{1.8062} \\ &\quad \overline{4.5686} \\ 2 \log 31 &= 2.9828 \\ \log f &= \overline{1.5858}; \text{ whence } f = 38.53.\end{aligned}$$

36.

$$\begin{aligned}n \log x + \log y &= 8 + \log 8.7 \\ n \log 73.96 + \log 27.25 &= 8 + \log 8.7 = 8.9395 \\ &\quad \log 27.25 = 1.4354 \\ \therefore n \log 73.96 &= \overline{7.5041}.\end{aligned}$$

$$\begin{aligned}\therefore n &= \frac{7.5041}{\log 73.96} = \frac{7.5041}{1.8690} & 18.69) \overline{7504.1} (4.015 \\ &= 4.015. & &\quad \overline{281} \\ && &\quad \overline{94} \\ && &\quad \overline{1}\end{aligned}$$

37. $r^3 = \frac{3V}{4\pi} = \frac{3 \times 33.87}{4 \times 3.1416}.$

$$\begin{array}{rcl} \log 3 = .4771 & \log 4 = .6021 \\ \log 33.87 = 1.5298 & \log 3.1416 = .4971 \\ \hline & & \hline \\ & 2.0069 & 1.0992 \\ & \hline & 1.0992 \\ & 3) \quad .9077 & \\ & \hline & \end{array}$$

$$\log r = .3026; \text{ whence } r = 2.007.$$

38. Let d be the diameter; then

$$\frac{4}{3}\pi \left(\frac{d}{2}\right)^3 = (36.4)^3;$$

$$\therefore d^3 = \frac{6 \times (36.4)^3}{\pi};$$

$$\therefore 3 \log d = \log 6 + 3 \log 36.4 - \log \pi,$$

$$\text{Thus } d = 45.16 \text{ cm.}$$

$$\begin{array}{rcl} \log 6 = .7782 & & \\ 3 \log 36.4 = 4.6833 & & \\ \hline & 5.4615 & \\ \log \pi = .4971 & & \\ \hline 3) \quad 4.9644 & = 3 \log d & \\ \hline & 1.6548 & \end{array}$$

$$\text{antilog } 1.6548 = 45.16.$$

39. $2 \log v = \log r + \log g - \log 289$

$$= \log 4000 + \log 32.2 - \log 5280 - \log 289.$$

$$\log 4000 = 3.6021$$

$$\log 5280 = 3.7226$$

$$\log 32.2 = 1.5079$$

$$\log 289 = 2.4609$$

$$\begin{array}{r} 5.1100 \\ 6.1835 \\ \hline \end{array}$$

$$2) \overline{2.9265}$$

$$\log v = 1.4632; \text{ whence } v = .2905.$$

Let $E = \frac{2\pi r}{v \times 60^2}$; then $\log E = \log 2\pi r - \log v - 2 \log 60$

$$\log 2 = .3010$$

$$\log v = 1.4632$$

$$\log \pi = .4971$$

$$2 \log 60 = 3.5564$$

$$\log r = 3.6021$$

$$\begin{array}{r} 4.4002 \\ 3.0196 \\ \hline \end{array}$$

$$\begin{array}{r} 4.4002 \\ 3.0196 \\ \hline \end{array}$$

$$\log E = \overline{1.3806}; \text{ whence } E = 24, \text{ approximately.}$$

EXAMPLES. XV. e. PAGE 163 H.

In Examples 16—19 let the expression be denoted by x .

16. $\log \sin 27^\circ 13' = \overline{1.6602}$

17. $\log \sin 47^\circ 13' = \overline{1.8656}$

$$\log \cos 46^\circ 16' = \overline{1.8397}$$

$$\log \tan 22^\circ 27' = \overline{1.6162}$$

$$\begin{array}{l} \log x = \overline{1.4999}; \\ \text{whence } x = .3161. \end{array}$$

$$\begin{array}{l} \log x = \overline{.2494}; \\ \text{whence } x = 1.776. \end{array}$$

18. $\log \sin 34^\circ 17' = \overline{1.7507}$

19. $x = \cos 28^\circ 14' \times \cos 37^\circ 26'.$

$$\log \tan 82^\circ 6' = \overline{.8577}$$

$$\log \cos 28^\circ 14' = \overline{1.9450}$$

$$\begin{array}{r} .6084 \\ \hline \end{array}$$

$$\log \cos 37^\circ 26' = \overline{1.8998}$$

$$\log x = \overline{.6190};$$

$$\begin{array}{r} \overline{1.8448}; \\ \text{whence } x = .6995. \end{array}$$

$$\text{whence } x = 4.159.$$

$$20. \quad 7 \log \tan x = \log 11 - \log 13.$$

$$\begin{array}{r} \log 11 = 1.0414 \\ \log 13 = 1.1139 \\ \hline 7) \overline{1.9275} \end{array}$$

$$\log \tan x = 1.9896;$$

$$\text{whence } x = 44^\circ 19'.$$

$$\begin{aligned}
 & \text{(i)} \quad na^2 \cot \frac{\pi}{n} = 32 \cot 22\frac{1}{2}^\circ \\
 & \qquad \qquad \qquad = 32 \tan 67\frac{1}{2}^\circ \\
 & \qquad \qquad \qquad = 32 \times 19.3136 \\
 & \qquad \qquad \qquad = 77.2544. \\
 & \text{(ii)} \quad \frac{nr^2}{2} \sin \frac{2\pi}{n} = 5 \times (3.3)^2 \sin 36^\circ. \\
 & \qquad \qquad \qquad \log 5 = .6990 \\
 & \qquad \qquad \qquad 2 \log 3 = 1.0370 \\
 & \qquad \qquad \qquad \log \sin 36^\circ = \overline{1.7692} \\
 & \qquad \qquad \qquad \overline{1.5052}; \\
 & \therefore \text{required value} = 32.00.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \tan \phi &= \frac{.7}{1 - (.35)^2} \sin 56^\circ 14' \\
 &= \frac{.7}{1.35 \times .65} \sin 56^\circ 14'. \\
 \log .7 &= \bar{1}.8451 & \log 1.35 &= .1303 \\
 \log \sin 56^\circ 14' &= \bar{1}.9198 & \log .65 &= \bar{1}.8129 \\
 &\underline{-} & &\underline{-} \\
 & \bar{1}.7649 & & \bar{1}.9432 \\
 &\underline{-} & &\underline{-} \\
 & \bar{1}.8217; \text{ whence } \phi = 33^\circ 33'.
 \end{aligned}$$

$$l = \frac{2 \times 32.78 \times 19.23}{52.01} \cos 57^\circ 47'.$$

$$\log 32.78 = 1.5156$$

$$\log 38.46 = 1.5850$$

$$\log \cos 57^\circ 47' = \overline{1.7268}$$

$$\qquad\qquad\qquad \overline{2.8274}$$

$$\log 52.01 = \overline{1.7161}$$

$$\log l = \overline{1.1113}; \text{ whence } l = 12.92$$

$$25. \text{ Expression} = \frac{2 \times (48)^2 \sin 23^\circ}{32 \cdot 19 \times (63)^2 \cos^2 23^\circ}.$$

$\log 2 = .3010$ $\log 32 \cdot 19 = 1 \cdot 5077$
 $2 \log 48 = 3 \cdot 3624$ $2 \log .63 = 1 \cdot 5986$
 $\log \sin 23^\circ = 1 \cdot 5919$ $2 \log \cos 23^\circ = 1 \cdot 9280$
 $\overline{3 \cdot 2553}$ $\overline{1 \cdot 0343}$
 $1 \cdot 0343$
 $\overline{2 \cdot 2210};$ whence value of expression = 166.3.

EXAMPLES. XVI. a. PAGE 166.

$$1. \text{ First side} = \frac{s(s-a)}{c} + \frac{s(s-b)}{c} = \frac{s(2s-a-b)}{c} = s.$$

$$2. \text{ First side} = s \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ = s \sqrt{\frac{(s-a)^2}{s^2}} = s - a.$$

$$3. \text{ First side} = \frac{1 - \cos A}{1 + \cos B} = \frac{2 \sin^2 \frac{A}{2}}{2 \sin^2 \frac{B}{2}} = \frac{(s-b)(s-c)}{bc} \times \frac{ca}{(s-c)(s-a)} \\ = \frac{a(s-b)}{c(s-a)} = \frac{a(a+c-b)}{b(b+c-a)}.$$

$$4. \text{ First side} = \frac{b(s-b)(s-c)}{bc} + \frac{a(s-c)(s-a)}{ca} \\ = \frac{(s-c)\{s-b+s-a\}}{c} = \frac{c(s-c)}{c} = s - c.$$

$$5. \text{ Each of the expressions reduces to } \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$6. \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \times 14}{24 \times 7}} = \frac{1}{2}.$$

$$7. \cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{21 \times 6}{8 \times 7}} = \frac{3}{2}.$$

$$8. \text{ First side} = \frac{s(s-a) + s(s-b) + s(s-c)}{abc} \\ = \frac{s\{3s - (a+b+c)\}}{abc} = \frac{s^2}{abc}.$$

$$9. \text{ First side} = \frac{b-c}{a} \cdot \frac{s(s-a)}{bc} + \text{two similar terms} \\ = \frac{(b-c)(s^2 - as)}{abc} + \text{two similar terms} \\ = \frac{s^2\{(b-c) + (c-a) + (a-b)\} - s\{a(b-c) + b(c-a) + c(a-b)\}}{abc} \\ = 0.$$

EXAMPLES. XVI. b. PAGE 169.

$$1. \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{7 \times 4}{5 \times 8}} = \sqrt{\frac{7}{10}}.$$

$$\log \sin \frac{C}{2} = \frac{1}{2} (\log 7 - 1)$$

$$= 1.9225490$$

$$\log \sin 56^\circ 47' = \frac{1.9225205}{285}$$

$$\text{prop'l. increase} = \frac{285}{827} \times 60'' = 20.6'';$$

$$\therefore \frac{C}{2} = 56^\circ 47' 20.6'', \text{ and } C = 113^\circ 34' 41''.$$

$$\begin{array}{r} 285 \\ 827) 17100 \\ \underline{1654} \\ 5600 \\ 4962 \end{array} \quad \begin{array}{r} 60 \\ 20.6 \\ \hline 1654 \\ 5600 \\ 4962 \end{array}$$

$$2. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{16 \times 24}{67 \times 27}} = \sqrt{\frac{128}{603}}.$$

$$\log \tan \frac{A}{2} = 1.6634464$$

$$= \log \tan 24^\circ 44' 13'';$$

$$\therefore A = 49^\circ 28' 26''.$$

$$\begin{array}{r} \log 128 = 2.1072100 \\ \log 603 = 2.7803173 \\ \hline 2 \overline{) 1.3268927} \\ 1.6634464 \end{array}$$

$$3. \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{15 \times 5}{2 \times 2 \times 4 \times 6}} = \frac{5}{4\sqrt{2}}.$$

$$\log \cos \frac{B}{2} = \log 5 - \frac{5}{2} \log 2$$

$$= 1.9463950$$

$$\log \cos 27^\circ 53' = \frac{1.9464040}{90}$$

$$\text{prop'l. increase} = \frac{90}{669} \times 60'' = 8.07'';$$

$$\therefore \frac{B}{2} = 27^\circ 53' 8.07'', \text{ and } B = 55^\circ 46' 16''.$$

$$\begin{array}{r} \log 5 = .6989700 \\ \frac{5}{2} \log 2 = .7525750 \\ \hline 1.9463950 \end{array}$$

$$\begin{array}{r} 30 \\ 223) 1800 (8.07 \\ \underline{1784} \\ 1600 \end{array}$$

$$4. \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{9 \times 2}{5 \times 6}} = \sqrt{\frac{6}{10}}.$$

$$\begin{aligned}\log \cos \frac{C}{2} &= \frac{1}{2} (\log 6 - 1) \\ &= \bar{1}.8890757\end{aligned}$$

$$\log \cos 39^\circ 14' = \frac{\bar{1}.8890644}{113}$$

$$\text{diff.} \quad \underline{\hspace{2cm}}$$

$$\text{prop}^l. \text{ decrease} = \frac{113}{1032} \times 60'' = 6.6'';$$

$$\therefore \frac{C}{2} = 39^\circ 13' 53.4'', \text{ and } C = 78^\circ 27' 47''. \quad \left| \begin{array}{c} \frac{113}{1032} \\ \frac{60}{6192} \\ \frac{5880}{6192} \\ \underline{\hspace{2cm}} \end{array} \right.$$

$$5. \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{7.5 \times 2}{3.75}} = \sqrt{\frac{2^2}{10}}.$$

$$\begin{aligned}\log \tan \frac{C}{2} &= \frac{1}{2} \{2 \log 2 - 1\} = \frac{1}{2} (\bar{1}.6020600) \\ &= \bar{1}.8010300\end{aligned}$$

$$\log \tan 32^\circ 18' = \frac{\bar{1}.8008365}{1935}$$

$$\text{diff.} \quad \underline{\hspace{2cm}}$$

$$\text{prop}^l. \text{ increase} = \frac{1935}{2796} \times 60'' = 41.5''; \quad \left| \begin{array}{c} \frac{1935}{2796} \\ \frac{60}{11184} \\ \frac{4260}{2796} \\ \underline{\hspace{2cm}} \end{array} \right.$$

$$\therefore \frac{C}{2} = 32^\circ 18' 41.5'', \text{ and } C = 64^\circ 37' 23''. \quad \left| \begin{array}{c} \frac{41.5}{14640} \\ \underline{\hspace{2cm}} \end{array} \right.$$

$$6. \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{3 \times 4}{15 \times 8}} = \sqrt{\frac{1}{10}}.$$

$$\log \tan \frac{C}{2} = -\frac{1}{2} = \bar{1}.5000000$$

$$\log \tan 17^\circ 33' = \frac{\bar{1}.500042}{42}$$

$$\text{prop}^l. \text{ decrease} = \frac{42}{439} \times 60'' = 5.7''; \quad \left| \begin{array}{c} \frac{42}{439} \\ \frac{60}{2195} \\ \underline{\hspace{2cm}} \end{array} \right.$$

$$\therefore \frac{C}{2} = 17^\circ 32' 54.3'', \text{ and } C = 35^\circ 5' 49''. \quad \left| \begin{array}{c} \frac{5.7}{3250} \\ \underline{\hspace{2cm}} \end{array} \right.$$

7. Let $a=4$, $b=10$, $c=11$.

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{25}{2} \cdot \frac{3}{2} \cdot \frac{1}{40}} = \sqrt{\frac{15}{2^5}} = \sqrt{\frac{30}{2^6}}.$$

$$\log \cos \frac{C}{2} = \frac{1}{2} (\log 3 + 1 - 6 \log 2) \\ = \bar{1}.8354707$$

$$\log \cos 46^\circ 47' = \frac{\bar{1}.8355378}{\text{diff. } 671}$$

$$\text{prop'l. increase} = \frac{671}{1345} \times 60'' = 30''.$$

$$\therefore \frac{C}{2} = 46^\circ 47' 30'', \text{ and } C = 93^\circ 35'.$$

$$\begin{array}{rcl} 1 + \log 3 & = & 1.4771213 \\ 6 \log 2 & = & 1.8061800 \\ \hline 2) & \bar{1}.6709413 \\ \hline & \bar{1}.8354707 \end{array}$$

8. $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{63 \times 7}{21 \times 8}} = \frac{1}{2}.$

$$\log \tan \frac{B}{2} = -\log 2 = \bar{1}.6989700$$

$$\log \tan 26^\circ 33' = \frac{\bar{1}.6986847}{2853}$$

$$\text{prop'l. increase} = \frac{2853}{3159} \text{ of } 60'' = 54.2''.$$

$$\therefore \frac{B}{2} = 26^\circ 33' 54.2'', \text{ and } B = 53^\circ 7' 48''.$$

$$\begin{array}{rcl} 2853 & & 60 \\ 3159) & \bar{1}.71180 & (54.2 \\ 15795 & & \\ \hline 13230 & & \\ 12636 & & \\ \hline 5940 & & \end{array}$$

Again $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{6 \times 8}{21 \times 7}} = \frac{4}{7}.$

$$\log \tan \frac{C}{2} = 2 \log 2 - \log 7 \\ = \bar{1}.7569620$$

$$\begin{array}{rcl} 2 \log 2 & = & .6020600 \\ \log 7 & = & .8450980 \\ \hline & & \bar{1}.7569620 \end{array}$$

$$\log \tan 29^\circ 44' = \frac{\bar{1}.7567587}{2033}$$

$$\text{prop'l. increase} = \frac{2033}{2933} \times 60'' = 41.5''.$$

$$\therefore \frac{C}{2} = 29^\circ 44' 41.5'', \text{ and } C = 59^\circ 29' 23''.$$

$$\therefore A = 67^\circ 22' 49''.$$

$$\begin{array}{rcl} 2033 & & 60 \\ 2933) & \bar{1}.21980 & (41.5 \\ 11732 & & \\ \hline 4660 & & \\ 2933 & & \\ \hline 17270 & & \end{array}$$

$$9. \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{3}{2} \times \frac{5}{2} \times \frac{2}{9} \times 2} = \sqrt{\frac{5}{3}}.$$

$$\begin{aligned}\log \tan \frac{B}{2} &= \frac{1}{2}(\log 5 - \log 3) \\ &= 1.109244 \\ \log \tan 52^\circ 14' &= 1.108395 \\ \text{diff.} & \quad 849\end{aligned}$$

$$\begin{array}{r|l} \log 5 & = .6989700 \\ \log 3 & = .4771213 \\ 2 & | \underline{.2218487} \\ & \cdot1109244 \end{array}$$

$$\text{prop'l. increase} = \frac{849}{435} \times 10'' = 19.5'';$$

$$\therefore \frac{B}{2} = 52^\circ 14' 19.5'', \text{ and } B = 104^\circ 28' 39''.$$

$$\text{Again } \tan \frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\frac{1}{2} \times \frac{3}{2} \times \frac{2}{9} \times \frac{2}{5}} = \frac{1}{\sqrt{3 \times 5}}.$$

$$\begin{aligned}\log \tan \frac{C}{2} &= -\frac{1}{2}(\log 3 + \log 5) \\ &= \bar{1}.4119544\end{aligned}$$

$$\begin{aligned}\log \tan 14^\circ 28' &= \bar{1}.4116146 \\ \text{diff.} & \quad 3398\end{aligned}$$

$$\text{prop'l. increase} = \frac{3398}{870} \times 10'' = 39' ;$$

$$\therefore \frac{C}{2} = 14^\circ 28' 39'', \text{ and } C = 28^\circ 57' 18''.$$

$$\therefore A = 46^\circ 34' 3''.$$

EXAMPLES. XVI. c. PAGE 173.

$$1. \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{5} \cot 30^\circ = \frac{2}{10} \sqrt{3}.$$

$$\begin{aligned}\log \tan \frac{A-B}{2} &= \log 2 + \frac{1}{2} \log 3 - 1 \\ &= \bar{1}.5395907\end{aligned}$$

$$\begin{aligned}\log \tan 19^\circ 6' &= \bar{1}.5394287 \\ \text{diff.} & \quad 1620\end{aligned}$$

$$\begin{array}{r|l} \log 2 & = .3010300 \\ \frac{1}{2} \log 3 & = .2385607 \\ & | \underline{.5395907} \end{array}$$

$$\text{prop'l. increase} = \frac{1620}{4084} \times 60'' = 24''.$$

$$\therefore \frac{A-B}{2} = 19^\circ 6' 24'', \text{ and } \frac{A+B}{2} = 60^\circ;$$

$$\therefore A = 79^\circ 6' 24''; B = 40^\circ 53' 36''.$$

2. $\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{8}{10} \cot 32^\circ 30'.$

$$\log \tan \frac{C-A}{2} = 3 \log 2 - 1 + \log \cot 32^\circ 30'$$

$$\begin{array}{rcl} & = .9989027 \\ \log \tan 51^\circ 28' & = & .9988763 \\ \text{diff.} & & 264 \end{array}$$

$$\text{prop'l. increase} = \frac{264}{2592} \times 60'' = 6'';$$

$$\therefore \frac{C-A}{2} = 51^\circ 28' 6'',$$

$$\frac{C+A}{2} = 57^\circ 33';$$

$$\therefore C = 108^\circ 58' 6''; A = 6^\circ 1' 54''.$$

$$\begin{array}{rcl} 3 \log 2 & = & .9030900 \\ \log \cot 32^\circ 30' & = & .1958127 \\ 1.0989027 & & \end{array}$$

3. $\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{5}{12} \sqrt{3}.$

$$\begin{array}{rcl} \log \tan \frac{B-A}{2} & = & 1 - 3 \log 2 - \frac{1}{2} \log 3 \\ & & = 1.8583494 \end{array}$$

$$\log \tan 35^\circ 49' = \frac{1.8583357}{137}$$

$$\text{prop'l. increase} = \frac{137}{2662} \times 60'' = 3'';$$

$$\therefore \frac{B-A}{2} = 35^\circ 49' 3'',$$

$$\frac{B+A}{2} = 60^\circ;$$

$$\therefore B = 95^\circ 49' 3''; A = 24^\circ 10' 57''.$$

$$\begin{array}{rcl} \frac{1}{2} \log 3 & = & .2385606 \\ 3 \log 2 & = & .9030900 \\ 1.1416506 & & \end{array}$$

$$4. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{4}{50} \cot 22^\circ 15' = \frac{8}{100} \cot 22^\circ 15'.$$

$$\log \tan \frac{B-C}{2} = \bar{1}.2912491$$

$$\log \tan 11^\circ 3' = \frac{\bar{1}.2906713}{5778}$$

$$\text{prop}^l. \text{ increase} = \frac{5778}{6711} \times 60'' = 52'';$$

$$\begin{aligned} 3 \log 2 - 2 &= \bar{2}.9030900 \\ \log \cot 22^\circ 15' &= \frac{\cdot 3881591}{\bar{1}.2912491} \end{aligned}$$

$$\therefore \frac{B-C}{2} = 11^\circ 3' 52'', \text{ and } \frac{B+C}{2} = 67^\circ 45';$$

$$\therefore B = 78^\circ 48' 52''; C = 56^\circ 41' 8''.$$

$$5. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{10}{32} \cot 17^\circ 21' 15''.$$

$$\begin{aligned} \log \tan \frac{C-A}{2} &= 1 + \log \cot 17^\circ 21' 15'' - 5 \log 2 \\ &= 1.5051500 - 1.5051500 \\ &= 0. \end{aligned}$$

$$\therefore \frac{C-A}{2} = 45^\circ, \text{ and } \frac{C+A}{2} = 72^\circ 38' 45'';$$

$$\therefore C = 117^\circ 38' 45''; A = 27^\circ 38' 45''.$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{4} \cot 30^\circ 15'.$$

$$\begin{aligned} \log \tan \frac{A-B}{2} &= -2 \log 2 + \log \cot 30^\circ 15' \\ &= \bar{1}.63214 \end{aligned}$$

$$\begin{aligned} \log \cot 30^\circ 15' &= \cdot 23420 \\ 2 \log 2 &= \cdot 60206 \\ &= \bar{1}.63214 \end{aligned}$$

$$\log \tan 23^\circ 13' = \bar{1}.63240$$

$$\text{diff. } \frac{26}{26}$$

$$\text{prop}^l. \text{ decrease} = \frac{26}{35} \times 60'' = 45'';$$

$$\therefore \frac{A-B}{2} = 23^\circ 12' 15'', \text{ and } \frac{A+B}{2} = 59^\circ 45';$$

$$\therefore A = 82^\circ 57' 15''; B = 36^\circ 32' 45''.$$

7. $\tan \frac{A-C}{2} = \frac{a-c}{a+c} \cot \frac{B}{2} = \frac{71}{283} \cot 28^\circ 14'.$

$$\log \tan \frac{A-C}{2} = \bar{1}.3556602$$

$$\log \tan 12^\circ 46' = \bar{1}.3552267$$

$$\text{diff. } 4335$$

$$\text{prop'l. increase} = \frac{4335}{5859} \times 60'' = 44'';$$

$$\log 71 = 1.8512583$$

$$\log \cot \frac{B}{2} = \frac{\bar{1}.2700705}{\bar{2}.1213288}$$

$$\log 583 = \frac{2.7656686}{1.3556602}$$

$$\therefore \frac{A-C}{2} = 12^\circ 46' 44'', \text{ and } \frac{A+C}{2} = 61^\circ 46';$$

$$\therefore A = 74^\circ 32' 44''; C = 48^\circ 59' 16''.$$

8. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{3}{5} \cot 32^\circ 30'.$

$$\log \tan \frac{B-C}{2} = \bar{1}.9739640$$

$$\log \tan 43^\circ 18' = \bar{1}.9742133$$

$$\text{diff. } 2493$$

$$\text{prop'l. decrease} = \frac{2493}{2531} \times 60'' = 59'';$$

$$\log 3 = .4771213$$

$$\log 5 = .6989700$$

$$\bar{1}.7781513$$

$$\log \cot 32^\circ 30' = \frac{.1958127}{\bar{1}.9739640}$$

$$\therefore \frac{B-C}{2} = 43^\circ 17' 1'', \text{ and } \frac{B+C}{2} = 57^\circ 30';$$

$$\therefore B = 100^\circ 47' 1''; C = 14^\circ 12' 59''.$$

9. Here $\cot \frac{A-B}{2} = \frac{a+b}{a-b} \tan \frac{C}{2} = 2 \tan \frac{C}{2}.$

$$\therefore \log \cot \frac{A-B}{2} = \log 2 + \log \tan 15^\circ 5' 2.5''$$

$$= \bar{1}.7316236$$

$$\log 2 = .3010300$$

$$\log \cot 61^\circ 41' = \bar{1}.7314436$$

$$\text{diff. } 1800$$

$$\log \tan 15^\circ 5' = \bar{1}.4305727$$

$$\text{prop'l. decrease} = \frac{1800}{504} \times 10'' = 35.7'';$$

$$\frac{2.5}{10} \times 838 = \frac{209}{\bar{1}.7316236}$$

$$\therefore \frac{A-B}{2} = 61^\circ 40' 24.3'', \text{ and } \frac{A+B}{2} = 74^\circ 54' 51.5'';$$

$$\therefore A = 136^\circ 35' 21.8''; B = 13^\circ 14' 33.2''.$$

EXAMPLES. XVI. d. PAGE 174.

1. Here $A = 180^\circ - 114^\circ 45' = 65^\circ 15'$.

$$c = \frac{a \sin C}{\sin A} = \frac{100 \sin 54^\circ 30'}{\sin 65^\circ 15'}.$$

$$\begin{aligned}\log c &= 1.9525317 \\ &= \log 89.646162; \\ \therefore c &= 89.646162.\end{aligned}$$

$$\begin{aligned}\log \sin 54^\circ 30' &= 1.9106860 \\ \log 100 &= 2 \\ 1.9106860 &\\ \log \sin 65^\circ 15' &= \frac{1.9581543}{1.9525317}\end{aligned}$$

$$2. a = \frac{c \sin A}{\sin C} = \frac{270 \sin 55^\circ}{\sin 60^\circ} = 270 \sin 55^\circ \times \frac{2}{\sqrt{3}}.$$

$$\begin{aligned}\therefore \log a &= 1 + 3 \log 3 + \log \sin 55^\circ - \frac{1}{2} \log 3 + \log 2 \\ &= 2.4071977 \\ \log 255.38 &= 2.4071869 \\ \text{diff.} &= \frac{108}{108} \\ \therefore \text{prop'l. increase} &= \frac{108}{170} \times .01 = .0064; \\ \therefore a &= 255.3864.\end{aligned}$$

$$\begin{aligned}\log 270 &= 2.4313639 \\ \log \sin 55 &= 1.9133645 \\ \log 2 &= .3010300 \\ \frac{1}{2} \log 3 &= .2385607 \\ &2.4071977\end{aligned}$$

$$3. c = \frac{b \sin C}{\sin B} = \frac{100 \sin 62^\circ 5'}{\sin 72^\circ 14'}.$$

$$\begin{aligned}\log c &= 1.96749 \\ &= \log 92.788; \\ \therefore c &= 92.788.\end{aligned}$$

$$\begin{aligned}\log \sin 62^\circ 5' &= 1.94627 \\ \log 100 &= 2 \\ 1.94627 &\\ \log \sin 72^\circ 14' &= \frac{1.97878}{1.96749}\end{aligned}$$

4. Here $A = 180^\circ - 148^\circ 40' = 31^\circ 20'$.

$$b = \frac{a \sin B}{\sin A} = \frac{102 \sin 70^\circ 30'}{\sin 31^\circ 20'}.$$

$$\begin{aligned}\log b &= 2.267 \\ &= \log 185; \\ \therefore b &= 185.\end{aligned}$$

$$\begin{aligned}\log 102 &= 2.009 \\ \log \sin 70^\circ 30' &= 1.974 \\ 1.974 &\\ \log \sin 31^\circ 20' &= \frac{1.716}{2.267}\end{aligned}$$

Again $c = \frac{a \sin C}{\sin A}$,

$$\begin{aligned}\log c &= 2.283 \\ &= \log 192; \\ \therefore c &= 192.\end{aligned}$$

$$\begin{aligned}\log 102 &= 2.009 \\ \log \sin 78^\circ 10' &= \overline{1.990} \\ &\quad \overline{1.999} \\ \log \sin 31^\circ 20' &= \overline{1.716} \\ &\quad \overline{2.283}\end{aligned}$$

5. Here $c = \frac{a \sin C}{\sin A} = \frac{123}{\sqrt{2} \sin 15^\circ 43'}$.

$$\begin{aligned}\log c &= 2.5066124 \\ \log 321.10 &= \overline{2.5066403} \\ \text{diff.} &= \overline{279} \\ \text{prop}^1. \text{decrease} &= \frac{279}{135} \times .01 = .02066.\end{aligned}$$

Thus $c = 321.0793$.

$$\frac{1}{2} \log 2 = .1505150$$

$$\begin{aligned}\log \sin 15^\circ 43' &= \overline{1.4327777} \\ &\quad \overline{1.5832927} \\ \log 123 &= \overline{2.0899051} \\ &\quad \overline{2.5066124}\end{aligned}$$

6. $a = \frac{b \sin A}{\sin B} = \frac{1006.62 \sin 44^\circ}{\sin 66^\circ}$.
 $\log 1006.62 = 3.0028656$
 $\log \sin 44^\circ = \overline{1.8417713}$
 $\quad \quad \quad \overline{2.8446369}$
 $\log \sin 66^\circ = \overline{1.9607302}$
 $\log a = \overline{2.8839067}$
 $\therefore a = 765.4321.$

$$\begin{aligned}c &= \frac{b \sin C}{\sin B} = \frac{1006.62 \sin 70^\circ}{\sin 66^\circ} \\ \log 1006.62 &= 3.0028656 \\ \log \sin 70^\circ &= \overline{1.9729858} \\ &\quad \overline{2.9758514} \\ \log \sin 66^\circ &= \overline{1.9607302} \\ \log c &= \overline{3.0151212} \\ \therefore c &= 1035.43.\end{aligned}$$

7. Here $A = \text{supplement of } 75^\circ 45'$;
 $\therefore b = \frac{1652 \sin 26^\circ 30'}{\sin 75^\circ 45'}$.
 $\log b = 2.8852436$
 $\log 767.80 = \overline{2.8852481}$
 $\text{diff.} \quad \quad \quad 45$
 $\text{prop}^1. \text{decrease} = \frac{45}{57} \times .01 = .008; \quad \therefore b = 767.792.$

$$\begin{aligned}\log 1652 &= 3.2180100 \\ \log \sin 26^\circ 30' &= \overline{1.6495274} \\ &\quad \overline{2.8675374} \\ \log \sin 73^\circ 45' &= \overline{1.9822938} \\ &\quad \overline{2.8852436}\end{aligned}$$

Again $c = \frac{1652 \sin 47^\circ 15'}{\sin 73^\circ 45'}$;
 $\log c = 3.1016030$
 $\log 1263.6 = \overline{3.1016096}$
 $\text{diff.} \quad \quad \quad 66$
 $\text{prop}^1. \text{decrease} = \frac{66}{344} \times .01 = .019; \quad \therefore c = 1263.58.$

$$\begin{aligned}\log 1652 &= 3.2180100 \\ \log \sin 47^\circ 15' &= \overline{1.8658868} \\ &\quad \overline{3.0838968} \\ \log \sin 73^\circ 45' &= \overline{1.9822938} \\ &\quad \overline{3.1016030}\end{aligned}$$

EXAMPLES. XVI. e. PAGE 176.

$$1. \quad \sin A = \frac{a \sin B}{b} = \frac{145}{178} \sin 41^\circ 10'.$$

$$\log \sin A = \bar{1}.7293399,$$

$\therefore A = 32^\circ 25' 35''.$

$$\log 145 = 2.1613680$$

$$\log \sin 41^\circ 10' = \bar{1}.8183919$$

$$\bar{1}.9797599$$

$$\log 178 = \frac{2.2504200}{\bar{1}.7293399}$$

$$2. \quad \sin B = \frac{b}{a} \sin A = \frac{127}{85} \sin 26^\circ 26'.$$

$$\log \sin B = \bar{1}.8228972,$$

$\therefore B = 41^\circ 41' 28'';$

$$\log 127 = 2.1038037$$

$$\log \sin 26^\circ 26' = \bar{1}.6485124$$

$$\bar{1}.7523161$$

$$\log 85 = \frac{1.9294189}{\bar{1}.8228972}$$

and since $a < b$, there is another value of B , namely,

$$B = 138^\circ 18' 32''.$$

$$3. \quad \sin B = \frac{b}{c} \sin C = \frac{4}{5} \sin 45^\circ = \frac{8}{10} \cdot \frac{1}{\sqrt{2}},$$

$$\log \sin B = 3 \log 2 - 1 - \frac{1}{2} \log 2.$$

$$\therefore \log \sin B = \bar{1}.7525750.$$

$$3 \log 2 = .9030900$$

$$\therefore B = 34^\circ 26'$$

$$1 + \frac{1}{2} \log 2 = \frac{1.1505150}{\bar{1}.7525750}$$

and $A = 100^\circ 34'$.

$$4. \quad \sin B = \frac{b}{a} \sin A = \frac{1706}{1405} \sin 40^\circ.$$

$$\log \sin B = \bar{1}.8923702$$

$$\log 1706 = 3.2319790$$

$$\log \sin 51^\circ 18' = \frac{\bar{1}.8923342}{360}$$

$$\log \sin 40^\circ = \frac{1.8080675}{3.0400465}$$

$$\text{prop'l. increase} = \frac{360}{1012} \times 60'' = 21''.$$

$$\log 1405 = \frac{3.1476763}{\bar{1}.8923702}$$

$\therefore B = 51^\circ 18' 21''$; but since $a < b$, there is another value of B , namely, $128^\circ 41' 39''$.

5. $\sin C = \frac{c}{b} \sin B = \frac{394}{573} \sin 112^\circ 4' = \frac{394}{573} \cos 22^\circ 4'.$

$\therefore \log \sin C = \bar{1}.8043030$

$\log 394 = 2.5954962$

$\log \sin 39^\circ 35' = \frac{\bar{1}.8042757}{273}$

$\log \sin 112^\circ 4' = \frac{\bar{1}.9669614}{2.5624576}$

prop^{l.} increase = $\frac{273}{1527} \times 60'' = 11'';$

$\log 573 = \frac{2.7581546}{\bar{1}.8043030}$

$\therefore C = 39^\circ 35' 11''$; and $A = 28^\circ 20' 49''$.

6. $\sin C = \frac{c}{b} \sin B = \frac{12}{8.4} \sin 37^\circ 36' = \frac{1}{7} \sin 37^\circ 36'.$

$\log \sin C = \bar{1}.9403352$

$\log \sin 37^\circ 36' = \bar{1}.7854332$

$\log \sin 60^\circ 39' = \frac{\bar{1}.9403381}{29}$

$\log .7 = \frac{\bar{1}.8450980}{\bar{1}.9403352}$

prop^{l.} decrease = $\frac{29}{711} \times 60'' = 2.4'$.

$\therefore C = 60^\circ 38' 58''$; but since $b < c$, there is another value of C , namely, $119^\circ 21' 2''$. Thus $A = 81^\circ 45' 2''$, or $23^\circ 2' 58''$.

7. (i) Here $\sin C = \frac{c \sin A}{a} = \frac{250}{125} \times \frac{1}{2} = 1.$

$\therefore C = 90^\circ$, and there is no ambiguity.

(ii) Here $\sin C = \frac{250}{200} \times \frac{1}{2} = \frac{5}{8}$, and since $a < c$ there will be two values of C satisfying the data.

(iii) Here $\sin C = \frac{125}{200} \times \frac{1}{2} = \frac{5}{16}$; but since $a > c$ there is only one solution.

From (ii) we have $\log \sin C = \log 5 - \log 8 = \bar{1}.79588$.

$\therefore C = 38^\circ 41'$, or $141^\circ 19'$; and $A = 111^\circ 19'$, or $8^\circ 41'$.

Now in the obtuse-angled triangle we have

$$b = \frac{a \sin B}{\sin A} = \frac{200 \sin 8^\circ 41'}{\sin 30^\circ}.$$

$\log b = 1.7809601$

$\log 200 = 2.3010300$

$\log 60.389 = \frac{1.7809578}{23}$

$\log \sin 8^\circ 41' = \bar{1}.1789001$

prop^{l.} increase = $\frac{23}{72} \times .001 = .0003.$

$\log \sin 30^\circ = \bar{1}.6989700$

$\therefore b = 60.3893.$

EXAMPLES. XVI. f. PAGE 180.

$$1. \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \cdot 4}{12 \cdot 5}} = \sqrt{\frac{1}{5}};$$

$$\therefore \log \tan \frac{B}{2} = -\frac{1}{2} \log 5 = -\frac{1}{2}(1 - \log 2)$$

$$= 1.6505150$$

$$\log \tan 24^\circ 5' = \frac{1.6502809}{2341}$$

diff.

$$\text{prop}^1. \text{ increase} = \frac{2341}{3390} \times 60'' = 41.4'';$$

$$\therefore \frac{B}{2} = 24^\circ 5' 41.4'', \text{ and } B = 48^\circ 11' 23''.$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{4 \cdot 5}{12 \cdot 3}} = \sqrt{\frac{5}{3^2}};$$

$$\therefore \log \tan \frac{C}{2} = \frac{1}{2}(1 - \log 2 - 2 \log 3)$$

$$= 1.8723637$$

$$\log \tan 36^\circ 41' = \frac{1.8721123}{2514}$$

$$\text{prop}^1. \text{ increase} = \frac{2514}{2637} \times 60'' = 57.2'';$$

$$\therefore \frac{C}{2} = 36^\circ 41' 57.2'', \text{ and } C = 73^\circ 23' 54'';$$

$$\therefore A = 58^\circ 24' 43''.$$

$$2. \cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2}$$

$$= \frac{512}{162} \tan 12^\circ = \frac{256}{81} \tan 12^\circ.$$

$$\therefore \log \cot \frac{A}{2} = 8 \log 2 - 4 \log 3 + \log \tan 12^\circ$$

$$= 1.8272293 = \log \cot 56^\circ 6' 27'';$$

$$\therefore \frac{A}{2} = 56^\circ 6' 27'', \text{ and } A = 112^\circ 12' 54''.$$

$$\therefore B+C = 67^\circ 47' 6'', \text{ and } B-C = 24^\circ;$$

$$\therefore B = 45^\circ 53' 33'', \text{ and } C = 21^\circ 53' 33''.$$

3. $\sin A = \frac{2}{7}$, if A is the less of the two acute angles.

$$\begin{aligned}\log \sin A &= \log 2 - \log 7 \\ &= \bar{1} \cdot 455932\end{aligned}$$

$$\log \sin 14^\circ 11' = \frac{\bar{1} \cdot 455921}{11}$$

$$\text{prop}^l. \text{ increase} = \frac{11}{110} \times 60'' = 6''.$$

$$\therefore A = 14^\circ 11' 6'';$$

$$\therefore B = 90^\circ - 14^\circ 11' 6'' = 75^\circ 48' 54''.$$

4. Here $a = 2183$, $A = 30^\circ 22'$, $B = 78^\circ 14'$, $C = 71^\circ 24'$.

$$b = \frac{a \sin B}{\sin A} = \frac{2183 \sin 78^\circ 14'}{\sin 30^\circ 22'}.$$

$$\log b = 3 \cdot 6260817$$

$$\log 4227 \cdot 4 = \frac{3 \cdot 6260733}{84}$$

$$\text{prop}^l. \text{ increase} = \frac{84}{103} \times .1 = .0815;$$

$$\therefore b = 4227 \cdot 4815.$$

$$\log 2183 = 3 \cdot 3390537$$

$$\log \sin B = \frac{\bar{1} \cdot 9907766}{3 \cdot 3298303}$$

$$\log \sin A = \frac{\bar{1} \cdot 7037486}{3 \cdot 6260817}$$

5. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{9} \cot 11^\circ 10'.$

$$\begin{aligned}\therefore \log \tan \frac{B-C}{2} &= \log \cot 11^\circ 10' - 2 \log 3 \\ &= 70465 - 95424 = \bar{1} \cdot 75041.\end{aligned}$$

$$\therefore \frac{B-C}{2} = 29^\circ 22' 26'', \text{ and } \frac{B+C}{2} = 78^\circ 50';$$

$$\therefore B = 108^\circ 12' 26''; C = 49^\circ 27' 34''.$$

Now $a = \frac{c \sin A}{\sin C};$

$$\therefore \log a = \log 2 + \log \sin 22^\circ 20' - \log \sin 49^\circ 27' 34''$$

$$= 30103 + \bar{1} \cdot 57977 - \bar{1} \cdot 88079$$

$$= .00001;$$

$$\therefore a = 1, \text{ approximately.}$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = .56234 \cot 29^\circ 21' 3''.$$

Now

$$\log \cot 29^\circ 21' = .250015$$

$$\log \cot 29^\circ 22'' = .249715$$

$$\text{diff. for } 60'' \qquad \overline{300}$$

$$\therefore \text{prop}^l. \text{ decrease for } 3'' = \frac{3}{60} \times 300 = 15;$$

$$\therefore \log \cot 29^\circ 21' 3'' = .250000.$$

$$\therefore \log \tan \frac{A-B}{2} = \bar{1}.75 + .25 = 0.$$

$$\therefore \tan \frac{A-B}{2} = 1, \text{ so that } \frac{A-B}{2} = 45^\circ.$$

$$\text{Also } \frac{A+B}{2} = 60^\circ 38' 57''; \text{ whence } A = 105^\circ 38' 57'', B = 15^\circ 38' 57''.$$

$$7. \quad \sin B = \frac{b \sin A}{a} = \frac{12 \sin 30^\circ}{9};$$

$$\therefore \log \sin B = 1.07918 + \bar{1}.69897 - .95424 = \bar{1}.82391;$$

$\therefore B = 41^\circ 48' 39''$ or $138^\circ 11' 21''$, both values being admissible since $a < b$.

$$\therefore C = 108^\circ 11' 21'' \text{ or } 11^\circ 48' 39''.$$

$$\text{Again } c = \frac{b \sin C_1}{\sin B_1} = \frac{12 \sin 108^\circ 11' 21''}{\sin 41^\circ 48' 39''};$$

$$\therefore \log c = 1.07918 + \bar{1}.97774 - \bar{1}.82391 = 1.23301;$$

$$\therefore c = 17.1.$$

Similarly from $c = \frac{b \sin C_2}{\sin B_2}$, we easily obtain $c = 3.68$.

$$8. \quad \tan \frac{C}{2} = \frac{a-b}{a+b} \cot \frac{A-B}{2} = \frac{1}{2} \cot 45^\circ = \frac{1}{2};$$

$$\therefore \log \tan \frac{C}{2} = \log 1 - \log 2 = \bar{1}.6989700$$

$$\log \tan 26^\circ 33' \qquad \qquad \qquad = \bar{1}.6986847 \\ \text{diff.} \qquad \qquad \qquad \qquad \qquad \qquad \overline{2853}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2853}{3159} \times 60'' = 54.2'';$$

$$\therefore \frac{C}{2} = 26^\circ 33' 54.2'', \text{ and } C = 53^\circ 7' 48''.$$

Hence $\frac{A+B}{2} = 63^\circ 26' 6''$, and $\frac{A-B}{2} = 45^\circ$;

$$\therefore A = 108^\circ 26' 6'', \quad B = 18^\circ 26' 6''.$$

9. (1) Let $a = 1404$, $b = 960$, $A = 32^\circ 15'$;

then $\sin B = \frac{b \sin A}{a} = \frac{80}{117 \operatorname{cosec} 32^\circ 15'}$;

$$\begin{aligned}\therefore \log \sin B &= 1 + 3 \log 2 - (2 \log 3 + \log 13 + \log \operatorname{cosec} 32^\circ 15') \\ &= 1.5621316, \text{ on reduction.}\end{aligned}$$

$$\therefore B = 21^\circ 23'; \quad \therefore C = 126^\circ 22'.$$

(2) Let $a = 1404$, $b = 960$, $B = 32^\circ 15'$;

then $\sin A = \frac{117}{80 \operatorname{cosec} 32^\circ 15'}$;

$$\begin{aligned}\therefore \log \sin A &= 2 \log 3 + \log 13 - (1 + 3 \log 2 + \log \operatorname{cosec} 32^\circ 15') \\ &= 1.8923236, \text{ on reduction.}\end{aligned}$$

$\therefore A = 51^\circ 18'$, or $128^\circ 42'$ since the solution is ambiguous.

$$\therefore C = 96^\circ 27', \text{ or } 19^\circ 3'.$$

10. We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1 - \frac{c}{b}}{1 + \frac{c}{b}} \cot \frac{A}{2}$

$$= \frac{1 - \cos \phi}{1 + \cos \phi} \cot \frac{A}{2} = \tan^2 \frac{\phi}{2} \cot \frac{A}{2},$$

where $\cos \phi = \frac{c}{b} = \frac{10}{11}$.

Hence $\log \cos \phi = 1 - \log 11 = 1.958607$;

$$\therefore \phi = 24^\circ 37' 12''.$$

Again $\log \tan \frac{B-C}{2} = 2 \log \tan \frac{\phi}{2} + \log \cot \frac{A}{2}$

$$= 2.677782 + .495800$$

$$= 1.173582.$$

$$\therefore \frac{B-C}{2} = 8^\circ 28' 56.5'', \text{ and } \frac{B+C}{2} = 72^\circ 17' 30'';$$

$$\therefore B = 80^\circ 46' 26.5'', \quad C = 63^\circ 48' 33.5''.$$

$$14. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}.$$

$\log 326 = 2.5132176$ $\log 199 = \underline{2.2988531}$ $\quad\quad\quad \underline{4.8120707}$ $\quad\quad\quad \underline{5.6436263}$ $\quad\quad\quad \underline{2) \bar{1.1684444}}$	$\log 976 = 2.9894498$ $\log 451 = \underline{2.6541765}$ $\quad\quad\quad \underline{5.6436263}$	$\log \tan \frac{A}{2} = \bar{1.5842222}$ $\log \tan 21^\circ 0' = \bar{1.5841774}$ $\text{diff.} \quad \quad \quad \underline{448}$
		Diff. for $20'' = 3775$; $\therefore \text{prop}^l. \text{ increase} = \frac{448}{3775} \times 60'' = 7'';$
		$\therefore \frac{A}{2} = 21^\circ 0' 7'', \text{ and } A = 42^\circ 0' 14''.$

Again $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}};$

$\log 199 = 2.2988531$ $\log 451 = \underline{2.6541765}$ $\quad\quad\quad \underline{4.9530296}$ $\quad\quad\quad \underline{5.5026674}$ $\quad\quad\quad \underline{2) \bar{1.4503622}}$	$\log 976 = 2.9894498$ $\log 326 = \underline{2.5132176}$ $\quad\quad\quad \underline{5.5026674}$	$\log \tan \frac{B}{2} = \bar{1.7251811}$ $\log \tan 27^\circ 58' = \bar{1.7250646}$ $\text{diff.} \quad \quad \quad \underline{1165}$
		Diff. for $60'' = 3.049$; $\therefore \text{prop}^l. \text{ increase} = \frac{1165}{3049} \times 60'' = 23'';$
		$\therefore \frac{B}{2} = 27^\circ 58' 23'', \text{ and } B = 55^\circ 56' 46''.$
		$\therefore C = 82^\circ 3'.$

$$15. \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2\frac{1}{2} \times 1\frac{1}{2}}{5 \times 6}} = \sqrt{\frac{1}{8}};$$

$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2$ $= \bar{1.5484550},$	$\text{Diff. for } 1' = 3342;$ $\therefore \text{prop}^l. \text{ increase} = \frac{965}{3342} \times 60'' = 17.3'';$
$\log \sin 20^\circ 42' = \underline{\bar{1.5483585}}$ $\text{diff.} \quad \quad \quad \underline{965}$	
	$\therefore A = 41^\circ 24' 35''.$

16. Here $\sin C = \frac{c}{b} = \frac{28.58}{57.321}$.

$$\begin{array}{rcl} \log 28.58 & = & 1.4560622 \\ \log 57.321 & = & \underline{1.7583138} & \text{Diff. for } 60'' = 2196; \\ & & \bar{1}.6977484 & \therefore \text{prop'l. increase} = \frac{939}{2196} \times 60'' = 26''. \\ \log \sin 29^\circ 54' & = & \underline{\bar{1}.6976545} \\ \text{diff.} & & 939 \end{array}$$

$$\therefore C = 29^\circ 54' 26''; \text{ whence } A = 60^\circ 5' 34''.$$

17. Let C be the right angle, and $A = 18^\circ 37' 29''$; then

$$c = \frac{a}{\sin A} = \frac{284}{\sin 18^\circ 37' 29''}.$$

$$\begin{array}{rcl} \log 284 & = & 2.4533183 \\ \log \sin 18^\circ 37' 29'' & = & \bar{1}.5042917 \\ \log c & = & \underline{2.9490266} \\ \log 889.25 & = & \underline{2.9490239} \\ & & 27 \\ & 5 & \underline{25} \\ & & 20 \\ & 4 & \underline{20} \end{array} \quad \left| \quad \begin{array}{rcl} \log \sin 18^\circ 37' & = & \bar{1}.5041105 \\ 29 & \times 3748 & = & 1812 \\ 60'' & & & \\ \log \sin 18^\circ 37' 29'' & = & \underline{\bar{1}.5042917} \end{array} \right.$$

$$\therefore c = 889.2554 \text{ feet.}$$

18. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{8} \cdot \sqrt{3};$

$$\begin{array}{rcl} \therefore \log \tan \frac{B-C}{2} & = & \frac{1}{2} \log 3 - 3 \log 2 & \text{Diff. for } 60'' = 6112; \\ & & = \bar{1}.3354706 & \\ \log \tan 12^\circ 12' & = & \underline{\bar{1}.3348711} & \therefore \text{prop'l. increase} = \frac{5995}{6112} \times 60'' = 59''. \\ \text{diff.} & & 5995 & \\ & & & \therefore \frac{B-C}{2} = 12^\circ 12' 59'', \text{ and } \frac{B+C}{2} = 60^\circ; \end{array}$$

$$\therefore B = 72^\circ 12' 59'', \quad C = 47^\circ 47' 1''.$$

19. Let AC be the ladder, C the window, and B the foot of the wall; then from the right-angled triangle ABC ,

$$AC = b = \frac{42.37}{\sin 72^\circ 15'},$$

$$\log b = 1.6270585 - 1.9788175 \\ = 1.6482410$$

$$\log 44.487 = \frac{1.6482331}{\begin{array}{r} 79 \\ 8 \end{array}} \\ = 1.6482331 - 0.00125 \\ = 1.6479776$$

\therefore length of ladder = 44.4878 feet.

20. $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9.99}{53.91} \cot 17^\circ 30'.$

$$\log 9.99 = 0.9995655$$

$$\log \cot 17^\circ 30' = \frac{.5012777}{1.5008432}$$

Diff. for $60'' = 2892$;

$$\log 53.91 = \frac{1.7316693}{1.7691739}$$

$$\therefore \text{prop'l. increase} = \frac{1817}{2892} \times 60'' = 38'';$$

$$\log \tan 30^\circ 26' = \frac{1.7689922}{1817}$$

$$\text{diff. } \quad \quad \quad 1817$$

$$\therefore \frac{A-B}{2} = 30^\circ 26' 38'', \text{ and } \frac{A+B}{2} = 72^\circ 30';$$

$$\therefore A = 102^\circ 56' 38'', B = 42^\circ 3' 22''.$$

21. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11.29}{38.95} \cot 23^\circ 37' 30''.$

$$\log \cot 23^\circ 37' = 0.3592844$$

$$\text{Subtract } \frac{30}{60} \times 3441 \quad \frac{1721}{3591123}$$

$$\log 11.29 = \frac{1.0526939}{1.4118062} \quad \text{Diff. for } 60'' = 2743;$$

$$\log 38.95 = \frac{1.5905075}{2743} \quad \therefore \text{prop'l. increase} = \frac{2413}{2743} \times 60'' = 53''.$$

$$\log \tan \frac{B-C}{2} = 1.8212987$$

$$\log \tan 33^\circ 31' = \frac{1.8210574}{2413}$$

$$\therefore \frac{B-C}{2} = 33^\circ 31' 53'', \text{ and } \frac{B+C}{2} = 66^\circ 22' 30'';$$

$$\therefore B = 99^\circ 54' 23'', C = 32^\circ 50' 37''.$$

$$\text{Again } a = \frac{b \sin A}{\sin B} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 99^\circ 54' 23''} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\cos 9^\circ 54' 23''}.$$

$$\log \sin 47^\circ 15' = 1.8658868$$

$$\log 25 \cdot 12 = 1.4000196$$

$$1.2659064$$

$$1.9934760$$

$$\therefore \log a = \underline{1.2724304}$$

$$\log 18.725 = \underline{1.2724218}$$

$$86$$

$$4 \quad \underline{93}$$

$$\log \cos 9^\circ 54' = 1.9934844$$

$$\text{Subtract } \frac{23}{60} \times 220 =$$

$$84$$

$$1.9934760$$

$$\therefore a = 18.7254.$$

$$22. \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{1361 \cdot 12 \times 1024 \cdot 48}{1837 \cdot 2 \times 2173 \cdot 84}}.$$

$$\log 1361 \cdot 1 = 3.1338900$$

$$2 \quad \quad \quad 64$$

$$\log 1024 \cdot 4 = 3.0104696$$

$$8 \quad \quad \quad 339$$

$$6.1443999$$

$$6.6013840$$

$$2) \underline{1.5430159}$$

$$\log \sin \frac{B}{2} = 1.7715079$$

$$\text{Diff. for } 60'' = 1724;$$

$$\log \sin 36^\circ 13' = \frac{1.7714702}{377}$$

$$\therefore \text{prop'l. increase} = \frac{377}{1724} \times 60'' = 13''.$$

$$\therefore \frac{B}{2} = 36^\circ 13' 13'', \text{ and } B = 72^\circ 26' 26''.$$

$$23. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{6.4405 \times 14.9114}{52.1248 \times 30.7728}}.$$

$$\log 6.4405 = 0.8089196$$

$$\log 52.124 = 1.7170377$$

$$\log 14.911 = 1.1735068$$

$$8 \quad \quad \quad 66$$

$$4 \quad \quad \quad 116$$

$$\log 30.772 = 1.4881557$$

$$1.9824380$$

$$8 \quad \quad \quad 113$$

$$3.2052113$$

$$3.2052113$$

$$2) \underline{2.7772267}$$

$$\log \tan \frac{A}{2} = 1.3886134$$

$$\text{Diff. for } 60'' = 5475;$$

$$\log \tan 13^\circ 44' = \frac{1.3880837}{5297}$$

$$\therefore \text{prop'l. increase} = \frac{5297}{5475} \times 60'' = 58''.$$

$$\therefore \frac{A}{2} = 13^\circ 44' 58'', \text{ and } A = 27^\circ 29' 56''.$$

Again $\tan \frac{B}{2} = \sqrt{\frac{14.9114 \times 30.7728}{52.1248 \times 6.4405}}.$

$$\begin{array}{rcl} \log 14.9114 & = & 1.1735184 \\ \log 30.7728 & = & 1.4881670 \\ & & 2.6616854 \\ & & 2.5259639 \\ 2) \underline{1.357215} & & \\ \log \tan \frac{B}{2} & = & .0678608 \\ \log \tan 49^\circ 27' & = & \frac{.0677338}{1270} \end{array}$$

$$\begin{array}{l} \log 52.1248 = 1.7170443 \\ \log 6.4405 = .8089196 \\ 2.5259639 \end{array}$$

$$\therefore \text{Diff. for } 60'' = 2558; \quad \therefore \text{prop'l. increase} = \frac{1270}{2558} \times 60'' = 30''.$$

$$\therefore \frac{B}{2} = 49^\circ 27' 30'', \text{ and } B = 98^\circ 55'.$$

$$\therefore C = 53^\circ 35' 4''.$$

24. $\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{202.949}{1497.597} \cot 51^\circ 36' 27''.$

$$\begin{array}{rcl} \log \cot 51^\circ 36' & = & 1.8990487 \\ \text{Subtract } \frac{27}{60} \times 2595 & = & \frac{1168}{1.8989319} \\ & & 9 \qquad \qquad \qquad 261 \\ & & 7 \qquad \qquad \qquad 203 \\ \log 202.94 & = & 2.3073677 \\ & & \underline{9} \qquad \qquad \qquad \underline{193} \\ & & \underline{2.2063189} \\ & & \underline{3.1753949} \\ \log \tan \frac{C-B}{2} & = & \underline{1.0309240} \\ \log \tan 6^\circ 7' & = & \underline{1.0300464} \\ \text{diff.} & & \underline{8776} \end{array}$$

$$\begin{array}{rcl} \log 1497.5 & = & 3.1753668 \\ & & 9 \\ & & 7 \qquad \qquad \qquad 203 \\ & & \underline{3.1753949} \end{array}$$

$$\text{Diff. for } 60'' = 11909; \quad \therefore \text{prop'l. increase} = \frac{8776}{11909} \times 60'' = 44'';$$

$$\therefore \frac{C-B}{2} = 6^\circ 7' 44'', \text{ and } \frac{C+B}{2} = 38^\circ 23' 33'';$$

$$\therefore C = 44^\circ 31' 17'', B = 32^\circ 15' 49''.$$

To find a , we have $a = \frac{b \sin A}{\sin B} = \frac{647 \cdot 324 \sin 103^\circ 12' 54''}{\sin 32^\circ 15' 49''}$.

Now

$$\log \sin 103^\circ 12' 54'' = \log \sin 76^\circ 47' 6''.$$

$$\log \sin 76^\circ 47' = \bar{1} \cdot 9883415$$

$$\frac{6}{60} \times 297 = \frac{30}{1 \cdot 9883445}$$

$$\log 647 \cdot 32 = 2 \cdot 8111190$$

$$\begin{array}{r} 4 \\ 27 \\ \hline 2 \cdot 7994662 \end{array}$$

$$\bar{1} \cdot 7273911$$

$$\log a = 3 \cdot 0720751$$

$$\log 1180 \cdot 5 = \frac{3 \cdot 0720660}{91}$$

$$\begin{array}{r} 2 \\ 73 \\ \hline 180 \\ 5 \\ \hline 185 \end{array}$$

$$\log \sin 32^\circ 15' = \bar{1} \cdot 7272276$$

$$\frac{49}{60} \times 2002 = \frac{1635}{1 \cdot 7273911}$$

$$\therefore a = 1180 \cdot 525.$$

25. $a = \frac{b \sin A}{\sin B} = \frac{23 \cdot 2783 \sin 37^\circ 57'}{\sin 43^\circ 13'}$.

$$\log 23 \cdot 278 = 1 \cdot 3669457$$

$$\begin{array}{r} 3 \\ 56 \end{array}$$

$$\log \sin 37^\circ 57' = \frac{\bar{1} \cdot 7888565}{1 \cdot 1558078}$$

$$\log \sin 43^\circ 13' = \bar{1} \cdot 8355378$$

$$\log a = \frac{1 \cdot 3202700}{198}$$

$$\log 20 \cdot 905 = \frac{1 \cdot 3202502}{187}$$

$$\therefore a = 20 \cdot 9059.$$

Again $c = \frac{b \sin C}{\sin B} = \frac{23.2783 \sin 81^\circ 10'}{\sin 43^\circ 13'}.$

$$\log c = 1.3669503 + 1.9948181 - 1.8355378$$

$$= 1.5262316$$

$$\begin{array}{r} \log 33.591 = \underline{1.5262229} \\ \qquad \qquad \qquad 87 \\ \qquad \qquad \qquad 7 \\ \qquad \qquad \qquad \underline{90} \end{array}$$

$$\therefore c = 33.5917.$$

26. $b = \frac{c \sin B}{\sin C} = \frac{2484.3 \sin 72^\circ 43' 25''}{\sin 47^\circ 12' 17''}.$

$$\log \sin 72^\circ 43' = 1.9799339$$

$$\frac{25}{60} \times 393 = \underline{164}$$

$$\log 2484.3 = \underline{3.3952040}$$

$$\qquad \qquad \qquad \underline{3.3751543}$$

$$\qquad \qquad \qquad \underline{\bar{1}.8655693}$$

$$\log b = \underline{3.5095850}$$

$$\log 3232.8 = \underline{3.5095788}$$

$$\qquad \qquad \qquad \underline{62}$$

$$\qquad \qquad \qquad \underline{4}$$

$$\qquad \qquad \qquad \underline{54}$$

$$\qquad \qquad \qquad \underline{6}$$

$$\log \sin 47^\circ 12' = 1.8655362$$

$$\frac{17}{60} \times 1169 = \underline{331}$$

$$\underline{\bar{1}.8655693}$$

$$\therefore b = 3232.846.$$

Again $a = \frac{c \sin A}{\sin C} = \frac{2484.3 \sin 60^\circ 4' 18''}{\sin 47^\circ 12' 17''}.$

$$\log \sin 60^\circ 4' = 1.9378220$$

$$\frac{18}{60} \times 727 = \underline{218}$$

$$\log 2484.3 = \underline{3.3952040}$$

$$\qquad \qquad \qquad \underline{3.3330478}$$

$$\log \sin 47^\circ 12' 17'' = \underline{\bar{1}.8655693}$$

$$\qquad \qquad \qquad \underline{3.4674785}$$

$$\log 2934.1 = \underline{3.4674749}$$

$$\qquad \qquad \qquad \underline{36}$$

$$\qquad \qquad \qquad \underline{2}$$

$$\qquad \qquad \qquad \underline{4}$$

$$\qquad \qquad \qquad \underline{6}$$

$$\therefore a = 2934.124.$$

$$27. \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\begin{aligned}\log \tan \frac{C-B}{2} &= 3.6401832 + .5497060 - 3.6690378 \\ &= .5208514 \\ \log \tan 73^\circ 13' &= \frac{.5205681}{2833} \quad \text{Diff. for } 60'' = 4568; \\ \text{diff.} &\quad \therefore \text{prop}^l. \text{ increase} = \frac{2833}{4568} \times 60'' = 37''.\end{aligned}$$

$$\therefore \frac{C-B}{2} = 73^\circ 13' 37'', \text{ and } \frac{C+B}{2} = 74^\circ 15';$$

$$\therefore C = 147^\circ 28' 37'', B = 1^\circ 1' 23''.$$

Again $a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31' 23''}.$

$\log 4517$	$= 3.6548501$	$\log \sin 32^\circ 31' = 1.7304148$
$\log \sin 31^\circ 30'$	$= 1.7180851$	$\frac{23}{60} \times 1981 = 759$
	3.3729352	
$\log \sin 32^\circ 31' 23''$	$= 1.7304907$	$\log \sin 32^\circ 31' 23'' = 1.7304907$
$\log a$	$= 3.6424445$	
$\log 4389.8$	$= 3.6424447$	

$$\therefore a = 4389.8 \text{ nearly.}$$

$$28. \sin A = \frac{a \sin C}{c} = \frac{324.68 \sin 35^\circ 17' 12''}{421.73}.$$

$$\log \sin 35^\circ 17' = 1.7616424$$

$$\frac{12}{60} \times 1784 = 357$$

$$\log 324.68 = 2.5114555$$

$$\log 421.73 = 2.6250345$$

$$\log \sin A = 1.6480991$$

$$\log \sin 26^\circ 24' = 1.6480038$$

Diff. for $60'' = 2544;$

$$\text{diff.} \quad \therefore \text{prop}^l. \text{ increase} = \frac{953}{2544} \times 60'' = 23'';$$

$$\therefore A = 26^\circ 24' 23'', \text{ and } \therefore B = 118^\circ 18' 25''.$$

Again $b = \frac{c \sin B}{\sin C} = \frac{421.73 \sin 61^\circ 41' 35''}{\sin 35^\circ 17' 12''}$.

$$\log \sin 61^\circ 41' = 1.9446501$$

$$\frac{35}{60} \times 680 = 397$$

$$\log 421.73 = 2.6250345$$

$$\log \sin 35^\circ 17' 12'' = 1.7616781$$

$$2.5697243$$

$$\log 642.75 = 2.8080421$$

$$\begin{array}{r} 41 \\ 6 \quad \quad \quad 41 \\ \hline \end{array}$$

$$\therefore b = 642.756.$$

29. $\sin C = \frac{c \sin A}{a} = \frac{435.6 \sin 36^\circ 18' 27''}{321.7}$.

$$\log \sin 36^\circ 18' = 1.7723314$$

$$\frac{27}{60} \times 1719 = 774$$

$$\log 435.6 = 2.6390879$$

$$2.4114967$$

$$\log 321.7 = 2.5074511$$

$$\log \sin C = 1.9040456$$

Diff. for $60'' = 943$;

$$\log \sin 53^\circ 17' = 1.9039587$$

$$\text{diff. } 869$$

$$\therefore \text{prop'l. increase} = \frac{869}{943} \times 60'' = 55'';$$

$$\text{diff. } 869$$

$\therefore C = 53^\circ 17' 55''$, or $126^\circ 42' 5''$, both values being admissible since $a < c$.

30. $\sin C = \frac{c \sin B}{b} = \frac{1665}{1325} \sin 52^\circ 19'$.

$$\log \sin C = 3.2214142 + 1.8983968 - 3.1222159$$

$$= 1.9975951$$

$$\log \sin 83^\circ 58' = 1.9975877$$

$$\text{Diff. for } 60'' = 134;$$

$$\text{diff. } 74$$

$$\therefore \text{prop'l. increase} = \frac{74}{134} \times 60'' = 33''.$$

$\therefore C = 83^\circ 58' 33''$, or $96^\circ 1' 27''$, both values being admissible since $b < c$.

32. $\sin B = \frac{b}{c} \sin C = \frac{17}{12} \sin 43^\circ 12' 12''.$

$$\log 17 = 1.2304489$$

$$\log \sin 43^\circ 12' = \bar{1}.8354033$$

$$\frac{17}{60} \times 1345 = \underline{\quad 269 \\ 1.0658791}$$

$$\log 12 = \underline{1.0791812}$$

$$\log \sin B = \underline{\bar{1}.9866979}$$

$$\log \sin 75^\circ 53' = \underline{\bar{1}.9866827 \\ \text{diff.} \quad \quad \quad 152}$$

Diff. for $60'' = 317$;

$$\therefore \text{prop'l. increase} = \frac{152}{317} \times 60'' = 29''.$$

$\therefore B = 75^\circ 53' 29'',$ or $104^\circ 6' 31'',$ both values being admissible since $c < b.$
 $\therefore A = 60^\circ 54' 19'',$ or $32^\circ 41' 17''.$

33. Let $b = 2.7402,$ $c = .7401,$ $A = 59^\circ 27' 5''.$

$$\tan \frac{B-C}{2} = \frac{2.0001}{3.4803} \cot 29^\circ 43' 32.5''.$$

$$\log \cot 29^\circ 43' = .2435347$$

$$\frac{32.5}{60} \times 2934 = \underline{\quad 1589}$$

$$\log 2.0001 = \underline{.3010517 \\ .5444275}$$

$$\log 3.4803 = \underline{.5416167}$$

$$\log \tan \frac{B-C}{2} = \underline{.0028108}$$

Diff. for $60'' = 2527;$

$$\log \tan 45^\circ 11' = \underline{.0027793 \\ \text{diff.} \quad \quad \quad 315}$$

$$\therefore \text{prop'l. increase} = \frac{315}{2527} \times 60'' = 7.5''.$$

$$\therefore \frac{B-C}{2} = 45^\circ 11' 7.5'', \text{ and } \frac{B+C}{2} = 60^\circ 16' 27.5'';$$

$$\therefore B = 105^\circ 27' 35'', C = 15^\circ 5' 20''.$$

Again $a = \frac{c \sin A}{\sin C} = \frac{.7401 \sin 59^\circ 27' 5''}{\sin 15^\circ 5' 20''}.$

$$\log \sin 59^\circ 27' = \bar{1}.9350969$$

$$\log \sin 15^\circ 5' = \bar{1}.4153468$$

$$\frac{5}{60} \times 746 = \underline{\quad 62}$$

$$\frac{20}{60} \times 4684 = \underline{\quad 1561}$$

$$\log .7401 = \underline{1.8692904}$$

$$\underline{1.4155029}$$

$$\bar{1}.8043935$$

$$\bar{1}.4155029$$

$$\log a = \underline{.3888906}$$

$$\log 2.4484 = \underline{.3888824}$$

$$\underline{\quad 82 \\ 5 \quad \quad \quad 89}$$

$$\therefore a = 2.44845.$$

Let h = the altitude; then $h = b \sin C$;

$$\begin{aligned}\therefore \log h &= \log 2.7402 + \log \sin 15^\circ 5' 20'' \\ &= .4379823 + \bar{1}.4155029 \\ &= \bar{1}.8532852 \\ \log .71332 &= \frac{\bar{1}.8532844}{8} \\ 1 &\quad \underline{6} \\ \therefore \text{altitude} &= .713321.\end{aligned}$$

34. Let $b = 105.25$, $c = 76.75$, $B - C = 17^\circ 48''$;

then $\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2} = \frac{182}{28.5} \tan 8^\circ 54'$.

$$\begin{aligned}\log \cot \frac{A}{2} &= \log 182 + \log \tan 8^\circ 54' - \log 28.5 \\ &= 2.2600714 + \bar{1}.1947802 - 1.4548449 \\ &= .0000067 = \log \cot 45^\circ \text{ nearly.} \\ \therefore A &= 90^\circ \text{ nearly.}\end{aligned}$$

35. (1) $\sin C = \frac{c \sin A}{a} = \frac{36.5 \sin 43^\circ 15'}{20};$

$$\begin{aligned}\log \sin C &= 1.5622929 + \bar{1}.8358066 - 1.3010300 \\ &= .0970695,\end{aligned}$$

which is impossible, since $\sin C$ must be < 1 .

(2) $\sin C = \frac{36.5 \sin 43^\circ 15'}{30};$

$$\begin{aligned}\log \sin 43^\circ 15' &= \bar{1}.8358066 \\ \log 36.5 &= 1.5622929 \\ &\quad \underline{1.3980995} \\ \log 30 &= \frac{1.4771213}{\log \sin C} \\ &= \bar{1}.9209782\end{aligned}$$

Thus C is not a right angle, and since $a < c$ the solution is ambiguous.

(3) $\sin C = \frac{36.5 \sin 43^\circ 15'}{45}.$

$$\begin{aligned}\log \sin C &= 1.3980995 - 1.6532125 \\ &= \bar{1}.7448870 \quad \text{Diff. for } 60'' = 1890;\end{aligned}$$

$$\begin{aligned}\log \sin 33^\circ 45' &= \frac{\bar{1}.7447390}{1480} \quad \therefore \text{prop'l. increase} = \frac{148}{189} \times 60'' = 47''. \\ \text{diff.} &\end{aligned}$$

$$\therefore C = 33^\circ 45' 47''; \quad \therefore B = 102^\circ 59' 13''.$$

$$\text{Now } b = \frac{a \sin B}{\sin A} = \frac{45 \cos 12^\circ 59' 13''}{\sin 43^\circ 15'};$$

$$\log \cos 12^\circ 59' = \bar{1}.9887531$$

$$\begin{array}{r} \text{Subtract } \frac{13}{60} \times 292 = \\ \hline & 63 \\ & \bar{1}.9887468 \end{array}$$

$$\begin{array}{r} \log 45 \\ = 1.6532125 \\ \hline 1.6419593 \end{array}$$

$$\begin{array}{r} \log \sin 43^\circ 15' = \bar{1}.8358066 \\ \hline 1.8061527 \end{array}$$

$$\begin{array}{r} \log 63.996 \\ = \bar{1}.8061528 \end{array}$$

Thus $b = 63.996$.

36. For the first part of the Example, see Art. 197.

$$\tan \theta = \frac{2 \sqrt{17.32 \times 13.47}}{3.85} \sin 23^\circ 36' 30'';$$

$$\log \sin 23^\circ 36' = \bar{1}.6024388$$

$$\begin{array}{r} 30 \\ 60 \times 2890 = \\ \hline 1445 \end{array}$$

$$\begin{array}{r} \log 2 \\ = .3010300 \\ 1.1839578 \\ \hline 1.0875711 \end{array}$$

$$\log 3.85 = .5854607$$

$$\log \tan \theta = .5021104$$

$$\log \tan 72^\circ 31' = \frac{.5017184}{3920}$$

$$\log 17.32 = 1.2385479$$

$$\begin{array}{r} \log 13.47 = 1.1293676 \\ 2) 2.3679155 \\ \hline 1.1839578 \end{array}$$

$$\text{Diff. for } 60'' = 4410;$$

$$\therefore \text{prop'l. increase} = \frac{392}{441} \times 60'' = 53''.$$

$$\therefore \theta = 72^\circ 31' 53''.$$

Again $c = (a - b) \sec \theta$;

$$\log 3.85 = .5854607$$

$$\log \sec 72^\circ 31' = .5222591$$

$$\begin{array}{r} 53 \\ 60 \times 4013 = \\ \hline 3545 \end{array}$$

$$\begin{array}{r} \log 12.825 \\ = 1.1080574 \end{array}$$

$$\begin{array}{r} 169 \\ 5 \\ \hline 170 \end{array}$$

$$\therefore c = 12.8255.$$

37. See Art. 195.

$$\tan \phi = \frac{44.1}{10.5} \tan 22^\circ 36';$$

$$\log 44.1 = 1.6444386$$

$$\log \tan 22^\circ 36' = \overline{1.6193645}$$

$$\overline{1.2638031}$$

$$\log 10.5 = 1.0211893$$

$$\log \tan \phi = \overline{.2426138}$$

Diff. for $60'' = 2930$;

$$\log \tan 60^\circ 13' = \overline{.2423617}$$

$$\therefore \text{prop'l. increase} = \frac{2521}{2930} \times 60'' = 52''.$$

$$\text{diff.}$$

$$\therefore \phi = 60^\circ 13' 52''.$$

Again $c = (a - b) \cos \frac{C}{2} \sec \phi$;

$$\log 10.5 = 1.0211893$$

$$\log \cos 22^\circ 36' = \overline{1.9653006}$$

$$\log \sec 60^\circ 13' = \overline{.3038870}$$

$$\frac{52}{60} \times 2208 = \overline{1914}$$

$$\log c = \overline{1.2905683}$$

$$\log 19.523 = \overline{1.2905466}$$

$$\overline{217}$$

$$9 \quad \overline{200}$$

$$8 \quad \overline{170}$$

$$8 \quad \overline{178}$$

$$\therefore c = 19.52398.$$

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$$a = 25.5$$

(1.)

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{16.3 \times 9}{11.7 \times 19}}.$$

$$b = 11.7$$

$$c = 19$$

$$\log 16.3 = 1.2122$$

$$\log 11.7 = 1.0682$$

$$\log 9 = .9542$$

$$\log 19 = 1.2787$$

$$\overline{2.1664}$$

$$\overline{2.3469}$$

$$\overline{2.3469}$$

$$2 \overline{) 1.8195}$$

$$\log \sin \frac{A}{2} = \overline{1.9097}; \text{ whence } \frac{A}{2} = 54^\circ 19'.$$

$$\therefore A = 108^\circ 38'.$$

(2)

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{112.5 \times 19.25}{68.75 \times 63}}.$$

$$\begin{array}{rcl} \log 112.5 & = & 2.0511 \\ \log 19.25 & = & 1.2844 \\ & & \overline{3.3355} \\ & & 3.6366 \\ & 2) & \overline{1.6989} \end{array}$$

$$\begin{array}{rcl} \log 68.75 & = & 1.8373 \\ \log 63 & = & 1.7993 \\ & & \overline{3.6366} \end{array}$$

$$\log \cos \frac{B}{2} = \bar{1}.8495; \text{ whence } \frac{B}{2} = 45^\circ.$$

$$\therefore B = 90^\circ.$$

$$3. \quad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{5.72 \times 3.31}{11.24 \times 13.65}}.$$

$$\begin{array}{rcl} \log 5.72 & = & .7572 \\ \log 3.31 & = & .5198 \\ & & \overline{1.2772} \\ & & 2.1859 \\ & 2) & \overline{1.0913} \end{array}$$

$$\begin{array}{rcl} \log 11.24 & = & 1.0508 \\ \log 13.65 & = & 1.1351 \\ & & \overline{2.1859} \end{array}$$

$$\log \sin \frac{C}{2} = \bar{1}.5457; \text{ whence } \frac{C}{2} = 20^\circ 34'.$$

$$\therefore C = 41^\circ 8'.$$

(4)

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{1 \times 14}{23 \times 8}}.$$

$$\begin{array}{rcl} \log 14 & = & 1.1461 \\ & & 2.2648 \\ & 2) & \overline{2.8813} \end{array}$$

$$\begin{array}{rcl} \log 23 & = & 1.3617 \\ \log 8 & = & .9031 \\ & & \overline{2.2648} \end{array}$$

$$\log \tan \frac{A}{2} = \bar{1}.4407; \text{ whence } \frac{A}{2} = 15^\circ 25'. \quad \therefore A = 30^\circ 50'.$$

$$\text{Again, } \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 14}{23 \times 1}}.$$

$$\begin{array}{rcl} \log 8 & = & .9031 \\ \log 14 & = & 1.1461 \\ & & \overline{2.0492} \\ & & 1.3617 \\ & 2) & \overline{0.6875} \end{array}$$

$$\log 23 = 1.3617$$

$$\begin{array}{rcl} \log \tan \frac{B}{2} & = & .3438 \\ \log \tan 65^\circ 36' & = & .3433 \\ \text{diff.} & & \overline{5} \end{array}$$

Since 5 is the mean of the differences 3 and 7, the corresponding increase in the angle is 1.5.

$$\text{Hence } \frac{B}{2} = 65^\circ 37.5', \text{ and } B = 131^\circ 15'.$$

$$\therefore C = 17^\circ 55'.$$

$$\begin{array}{l} a = 68.7 \\ b = 93.2 \\ c = 63 \end{array}$$

$$\begin{array}{l} a = 15 \\ b = 22 \\ c = 9 \end{array}$$

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$$5. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{24.22 \times 7.32}{70.72 \times 39.18}}.$$

$$\begin{array}{rcl} \log 24.22 & = & 1.3842 \\ \log 7.32 & = & .8645 \\ & & | \\ & & 2.2487 \\ & & 3.4426 \\ 2) \underline{2.8061} & & \end{array} \quad \begin{array}{rcl} \log 70.72 & = & 1.8495 \\ \log 39.18 & = & 1.5931 \\ & & | \\ & & 3.4426 \end{array}$$

$$\tan \frac{A}{2} = \bar{1}.4031; \text{ whence } \frac{A}{2} = 14^\circ 12'. \quad \therefore A = 28^\circ 24'.$$

$$\text{Again,} \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{39.18 \times 7.32}{70.72 \times 24.22}}.$$

$$\begin{array}{rcl} \log 39.18 & = & 1.5931 \\ \log 7.32 & = & .8645 \\ & & | \\ & & 2.4576 \\ & & 3.2337 \\ 2) \underline{\bar{1}.2239} & & \end{array} \quad \begin{array}{rcl} \log 70.72 & = & 1.8495 \\ \log 24.22 & = & 1.3842 \\ & & | \\ & & 3.2337 \end{array}$$

$$\log \tan \frac{B}{2} = \bar{1}.6120; \text{ whence } \frac{B}{2} = 22^\circ 15'.$$

$$\therefore B = 44^\circ 30', \text{ and } C = 107^\circ 6'.$$

$$6. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{11.02 \times 22.38}{82.62 \times 49.22}}.$$

$$\begin{array}{rcl} \log 11.02 & = & 1.0422 \\ \log 22.38 & = & 1.3498 \\ & & | \\ & & 2.3920 \\ & & 3.6093 \\ 2) \underline{2.7827} & & \end{array} \quad \begin{array}{rcl} \log 82.62 & = & 1.9171 \\ \log 49.22 & = & 1.6922 \\ & & | \\ & & 3.6093 \end{array}$$

$$\tan \frac{A}{2} = \bar{1}.3914; \text{ whence } \frac{A}{2} = 13^\circ 50'. \quad \therefore A = 27^\circ 40'.$$

$$\text{Again,} \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{49.22 \times 22.38}{82.62 \times 11.02}}.$$

$$\begin{array}{rcl} \log 49.22 & = & 1.6922 \\ \log 22.38 & = & 1.3948 \\ & & | \\ & & 3.0420 \\ & & 2.9593 \\ 2) \underline{0.0827} & & \end{array} \quad \begin{array}{rcl} \log 82.62 & = & 1.9171 \\ \log 11.02 & = & 1.0422 \\ & & | \\ & & 2.9593 \end{array}$$

$$\tan \frac{B}{2} = \cdot 0414; \text{ whence } \frac{B}{2} = 47^\circ 43'.5, \text{ as in Ex. 4 above.}$$

$$\therefore B = 95^\circ 27'; \text{ and } C = 56^\circ 53'.$$

7. $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{15.98 \times 1.23}{6.84 \times 10.37}}$

log 15.98 =	1.2036
log 1.23 =	.0899
	1.2935
	1.8508
2)	1.4427

log 6.84 =	.8351
log 10.37 =	1.0157
	1.8508

$\log \cos \frac{A}{2} = -1.7214$; whence $\frac{A}{2} = 58^\circ 14'$. $\therefore A = 116^\circ 28'$.

8. $\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{10.85 \times 2.85}{21.5 \times 29.5}}$

log 10.85 =	1.0355
log 2.85 =	.4548
	1.4903
	2.8022
2)	2.6881

log 21.5 =	1.3324
log 29.5 =	1.4698
	2.8022

$\log \sin \frac{B}{2} = -1.3441$; whence $\frac{B}{2} = 12^\circ 45' .5$, as in Ex. 4 above.
 $\therefore B = 25^\circ 31'$.

9. $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$= \sqrt{\frac{3 \times 2}{10 \times 5}}$$

$$= \sqrt{.12}.$$

$\therefore \log \tan \frac{A}{2} = \frac{1}{2}(\bar{1}.0792)$
 $= \bar{1}.5396$;

whence $\frac{A}{2} = 19^\circ 6'$.

$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$$= \sqrt{\frac{5 \times 2}{10 \times 3}}$$

$$= \sqrt{\frac{1}{3}}.$$

$\therefore \log \tan \frac{B}{2} = \frac{1}{2}(-\log 3) = \frac{1}{2}(\bar{1}.5229)$
 $= \bar{1}.7614$;

whence $\frac{B}{2} = 30^\circ$.

Thus $A = 38^\circ 12'$, $B = 60^\circ$, $C = 81^\circ 48'$.

10. $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{7 \times 6}{21 \times 8}}$

log 7 =	.8451
log 6 =	.7782
	1.6233
	2.2253
2)	1.3980

log 21 =	1.3222
log 8 =	.9031
	2.2253

$\tan \frac{A}{2} = \bar{1}.6990$; whence $\frac{A}{2} = 26^\circ 34'$. $\therefore A = 53^\circ 8'$.

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 6}{21 \times 7}}$$

$$\log 8 = .9031$$

$$\log 6 = .7782$$

$$\overline{1.6813}$$

$$\overline{2.1673}$$

$$2) \overline{1.5140}$$

$$\log \tan \frac{B}{2} = \overline{1.7570}; \text{ whence } \frac{B}{2} = 29^\circ 45'.$$

$$\therefore B = 59^\circ 30', \text{ and } C = 67^\circ 22'.$$

$$11. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{17.8}{47.8} \cot 53^\circ 43'.$$

$$\log 17.8 = 1.2504$$

$$\log \cot 53^\circ 43' = \overline{1.8658}$$

$$\overline{1.1162}$$

$$\log 47.8 = \overline{1.6794}$$

$$\frac{B+C}{2} = 36^\circ 17',$$

$$\log \tan \frac{B-C}{2} = \overline{1.4368}; \quad \text{whence } \frac{B-C}{2} = 15^\circ 18'.$$

$$\therefore B = 51^\circ 35', \text{ and } C = 20^\circ 59'.$$

$$12. \quad \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{38.7}{232.1} \cot 61^\circ 51'.$$

$$\log 38.7 = 1.5877$$

$$\log \cot 61^\circ 51' = \overline{1.7284}$$

$$\overline{1.3161}$$

$$\log 232.1 = \overline{2.3657}$$

$$\frac{B+A}{2} = 28^\circ 9',$$

$$\log \tan \frac{B-A}{2} = \overline{2.9504}; \quad \text{whence } \frac{B-A}{2} = 5^\circ 6'.$$

$$\therefore B = 33^\circ 15', \text{ and } A = 23^\circ 3'.$$

$$13. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{.564}{.588} \cot 30^\circ 15'.$$

$$\log 564 = 2.7513$$

$$\log \cot 30^\circ 15' = .2342$$

$$\overline{2.9855}$$

$$\log 588 = \overline{2.7694}$$

$$\frac{C+A}{2} = 59^\circ 45',$$

$$\log \tan \frac{C-A}{2} = .2161; \quad \text{whence } \frac{C-A}{2} = 58^\circ 42'.$$

$$\therefore C = 118^\circ 27', \text{ and } A = 1^\circ 3'.$$

(14.) Here $a=27\cdot3$, $b=16\cdot8$, $C=45^\circ 7'$. Required A , B , and c .

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{10\cdot5}{44\cdot1} \cot 22^\circ 33'\cdot5.$$

$$\log 10\cdot5 = 1\cdot0212$$

$$\log \cot 22^\circ 33'\cdot5 = \underline{.3815}$$

$$\log 44\cdot1 = \underline{1\cdot6444}$$

$$\frac{A+B}{2} = 67^\circ 26'\cdot5,$$

$$\log \tan \frac{A-B}{2} = \underline{1\cdot7583}; \quad \text{whence } \frac{A-B}{2} = 29^\circ 49'.$$

$$\therefore A = 97^\circ 15'\cdot5, \text{ and } B = 37^\circ 37'\cdot5.$$

Again, $c = \frac{b \sin C}{\sin B} = \frac{16\cdot8 \sin 45^\circ 7'}{\sin 37^\circ 37'\cdot5}$.

$$\log 16\cdot8 = 1\cdot2253$$

$$\log \sin 45^\circ 7' = \underline{1\cdot8503}$$

$$\underline{1\cdot0756}$$

$$\log \sin 37^\circ 37'\cdot5 = \underline{1\cdot7857}$$

$$\log c = \underline{1\cdot2899}; \text{ whence } c = 19\cdot49.$$

15. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{57\cdot8}{108\cdot0} \cot 18^\circ 30'.$

$$\log 57\cdot8 = 1\cdot7619$$

$$\log \cot 18^\circ 30' = \underline{.4755}$$

$$\underline{2\cdot2374}$$

$$\log 108 = \underline{2\cdot0334}$$

$$\frac{B+C}{2} = 71^\circ 30',$$

$$\log \tan \frac{B-C}{2} = \underline{.2040}; \quad \text{whence } \frac{B-C}{2} = 57^\circ 59'.$$

$$\therefore B = 129^\circ 29', \text{ and } C = 13^\circ 31'.$$

Again, $a = \frac{c \sin A}{\sin C} = \frac{25\cdot1 \sin 37^\circ}{\sin 13^\circ 31'}$.

$$\log 25\cdot1 = 1\cdot3997$$

$$\log \sin 37^\circ = \underline{1\cdot7795}$$

$$\underline{1\cdot1792}$$

$$\log \sin 13^\circ 31' = \underline{1\cdot3687}$$

$$\log a = \underline{1\cdot8105}; \text{ whence } a = 64\cdot65.$$

16. $\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{6\cdot48}{60\cdot48} \cot 30^\circ.$

$$\log 6\cdot48 = .8116$$

$$\log \cot 30^\circ = \underline{.2386}$$

$$\underline{1\cdot0502}$$

$$\log 60\cdot48 = \underline{1\cdot7816}$$

$$\frac{C+B}{2} = 60^\circ,$$

$$\log \tan \frac{C-B}{2} = \underline{1\cdot2686};$$

$$\text{whence } \frac{C-B}{2} = 10^\circ 31'.$$

$$\therefore C = 70^\circ 31', \text{ and } B = 49^\circ 29'.$$

17. $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{8}{22} \cot 41^\circ 7' = \frac{4}{11} \cot 41^\circ 7'.$

$$\log 4 = \cdot 6021 \\ \log \cot 41^\circ 7' = \cdot 0590$$

$$\log 11 = \underline{\overline{1\cdot 0414}} \quad \frac{A+B}{2} = 48^\circ 53',$$

$$\log \tan \frac{A-B}{2} = \underline{\overline{1\cdot 6197}}; \quad \text{whence } \frac{A-B}{2} = 22^\circ 37'.$$

$\therefore A = 71^\circ 30', \text{ and } B = 26^\circ 16'.$

18. Let $a = 2b$, and $C = 52^\circ 47'$, then

$$\begin{aligned} \tan \frac{A-B}{2} &= \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{3} \cot 26^\circ 23' \cdot 5 \\ &= \frac{1}{3} \times 2 \cdot 0152 = \cdot 6717 = \tan 33^\circ 53'. \end{aligned}$$

$$\therefore \frac{A+B}{2} = 63^\circ 36' \cdot 5, \text{ and } \frac{A-B}{2} = 33^\circ 53'.$$

$\therefore A = 97^\circ 29' \cdot 5, \text{ and } B = 29^\circ 43' \cdot 5.$

19. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{44}{218} \cot 9^\circ 8'.$

$$\log 44 = \cdot 6435 \\ \log \cot 9^\circ 8' = \cdot 7938$$

$$\log 218 = \underline{\overline{2\cdot 3385}} \quad \frac{B+C}{2} = 80^\circ 52',$$

$$\log \tan \frac{B-C}{2} = \cdot 0988; \quad \text{whence } \frac{B-C}{2} = 51^\circ 28'.$$

$\therefore B = 132^\circ 20', \text{ and } C = 29^\circ 24'.$

20. $\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{6\cdot 43}{41\cdot 63} \cot 60^\circ 49'.$

$$\log 6\cdot 43 = \cdot 8082 \\ \log \cot 60^\circ 49' = \underline{\overline{1\cdot 7470}} \quad \frac{B+A}{2} = 29^\circ 11',$$

$$\log 41\cdot 63 = \underline{\overline{1\cdot 6194}} \quad \text{whence } \frac{B-A}{2} = 4^\circ 56'.$$

$\therefore B = 34^\circ 7', \text{ and } A = 24^\circ 15'.$

Again, $c = \frac{a \sin C}{\sin A} = \frac{17\cdot 6 \sin 121^\circ 38'}{\sin 24^\circ 15'} = \frac{17\cdot 6 \sin 58^\circ 22'}{\sin 24^\circ 15'}.$

$$\log 17\cdot 6 = 1\cdot 2455 \\ \log \sin 58^\circ 22' = \underline{\overline{1\cdot 9301}}$$

$$\log \sin 24^\circ 15' = \underline{\overline{1\cdot 6135}} \\ \log c = \underline{\overline{1\cdot 5620}}; \text{ whence } c = 36\cdot 48.$$

21.

$$a = \frac{b \sin A}{\sin B} = \frac{100 \sin 40^\circ}{\sin 70^\circ}.$$

$$\log 100 = 2$$

$$\log \sin 40^\circ = \overline{1.8081}$$

$$\log \sin 70^\circ = \overline{1.9730}$$

$$\log a = \overline{1.8351}; \text{ whence } a = 68.41.$$

b A E

22.

$$b = \frac{a \sin B}{\sin A} = \frac{85.2 \sin 42^\circ}{\sin 31^\circ}.$$

$$\log 85.2 = 1.9304$$

$$\log \sin 42^\circ = \overline{1.8255}$$

$$\log \sin 31^\circ = \overline{1.7118}$$

$$\log b = 2.0441; \text{ whence } b = 110.7.$$

23.

$$a = \frac{c \sin A}{\sin C} = \frac{5.23 \sin 49^\circ 11'}{\sin 109^\circ 34'} = \frac{5.23 \sin 49^\circ 11'}{\sin 70^\circ 26'}.$$

$$\log 5.23 = .7185$$

$$\log \sin 49^\circ 11' = \overline{1.8789}$$

$$\log \sin 70^\circ 26' = \overline{1.9742}$$

$$\log a = \overline{.6232}; \text{ whence } a = 4.200.$$

24.

$$c = \frac{b \sin C}{\sin B} = \frac{873 \sin 71^\circ 35'}{\sin 42^\circ 58'}.$$

$$\log 873 = 2.9410$$

$$\log \sin 71^\circ 35' = \overline{1.9772}$$

$$\overline{2.9182}$$

$$\log \sin 42^\circ 58' = \overline{1.8336}$$

$$\log c = \overline{3.0846}; \text{ whence } c = 1215.$$

25.

$$a = \frac{c \sin A}{\sin C} = \frac{60 \sin 60^\circ}{\sin 40^\circ 40'}.$$

$$\log 60 = 1.7782$$

$$\log \sin 60^\circ = \overline{1.9375}$$

$$\overline{1.7157}$$

$$\log \sin 40^\circ 40' = \overline{1.8140}$$

$$\log a = \overline{1.9017}; \text{ whence } a = 79.75.$$

$$26. \quad a = \frac{c \sin A}{\sin C} = \frac{3.57 \sin 51^\circ 51'}{\sin 40^\circ 26'}.$$

$$\log 3.57 = .5527$$

$$\log \sin 51^\circ 51' = \overline{1.8956}$$

$$\overline{.4483}$$

$$\log \sin 40^\circ 26' = \overline{1.8120}$$

$$\log a = .6363$$

$$\therefore a = 4.328.$$

$$b = \frac{c \sin B}{\sin C} = \frac{3.57 \sin 87^\circ 43'}{\sin 40^\circ 26'}.$$

$$\log 3.57 = .5527$$

$$\log \sin 87^\circ 43' = \overline{1.9996}$$

$$\overline{.5523}$$

$$\log \sin 40^\circ 26' = \overline{1.8120}$$

$$\log b = .7403$$

$$\therefore b = 5.499.$$

$$27. \quad b = \frac{a \sin B}{\sin A} = \frac{125.7 \sin 65^\circ 47'}{\sin 61^\circ 34'}.$$

$$\log 125.7 = 2.0993$$

$$\log \sin 65^\circ 47' = \overline{1.9600}$$

$$\overline{2.0593}$$

$$\log \sin 61^\circ 34' = \overline{1.9442}$$

$$\log b = \overline{2.1151}; \text{ whence } b = 130.3.$$

$$28. \quad a = \frac{c \sin A}{\sin C}$$

$$= \frac{92.93 \sin 72^\circ 19'}{\sin 24^\circ 24'}.$$

$$\log 92.93 = 1.9681$$

$$\log \sin 72^\circ 19' = \overline{1.9789}$$

$$\overline{1.9470}$$

$$\log \sin 24^\circ 24' = \overline{1.6161}$$

$$\log a = \overline{2.3309}$$

$$\therefore a = 214.2.$$

$$b = \frac{c \sin B}{\sin C}$$

$$= \frac{92.93 \sin 83^\circ 17'}{\sin 24^\circ 24'}.$$

$$\log 92.93 = 1.9681$$

$$\log \sin 83^\circ 17' = \overline{1.9970}$$

$$\overline{1.9651}$$

$$\log \sin 24^\circ 24' = \overline{1.6161}$$

$$\log b = \overline{2.3490}$$

$$\therefore b = 223.4.$$

$$29. \quad b = \frac{a \sin B}{\sin A}$$

$$= \frac{4.375 \sin 49^\circ 30'}{\sin 60^\circ}.$$

$$\log 4.375 = .6410$$

$$\log \sin 49^\circ 30' = \overline{1.8810}$$

$$\overline{.5220}$$

$$\log \sin 60^\circ = \overline{1.9375}$$

$$\log b = .5845$$

$$\therefore b = 3.841.$$

$$c = \frac{a \sin C}{\sin A}$$

$$= \frac{4.375 \sin 70^\circ 30'}{\sin 60^\circ}.$$

$$\log 4.375 = .6410$$

$$\log \sin 70^\circ 30' = \overline{1.9743}$$

$$\overline{.6153}$$

$$\log \sin 60^\circ = \overline{1.9375}$$

$$\log c = .6778$$

$$\therefore c = 4.762.$$

$$30. \quad \frac{A}{1} = \frac{B}{4} = \frac{C}{7} = \frac{A+B+C}{12} = \frac{180^\circ}{12} = 15^\circ.$$

$$\therefore A=15^\circ, B=60^\circ, C=105^\circ.$$

$$a = \frac{b \sin A}{\sin B} = \frac{89.36 \sin 15^\circ}{\sin 60^\circ}.$$

log 89.36 = 1.9512
 log sin 15° = 1.4130
1.3642
 log sin 60° = 1.9375
 log $a = \overline{1.4267}$
 ∴ $a = 26.71.$

$$c = \frac{b \sin C}{\sin B} = \frac{89.36 \sin 75^\circ}{\sin 60^\circ}.$$

log 89.36 = 1.9512
 log sin 75° = 1.9849
1.9361
 log sin 60° = 1.9375
 log $c = \overline{1.9986}$
 ∴ $c = 99.68.$

$$31. \quad \sin B = \frac{b \sin A}{a} = \frac{62 \sin 82^\circ 14'}{73}.$$

$$\begin{aligned} \log 62 &= 1.7924 \\ \log \sin 82^\circ 14' &= \overline{1.9960} \\ &\quad \overline{1.7884} \\ \log 73 &= \overline{1.8633} \\ \log \sin B &= \overline{1.9251}; \text{ whence } B = 57^\circ 18'. \end{aligned}$$

$$32. \quad \sin C = \frac{c \sin B}{b} = \frac{63.45 \sin 27^\circ 15'}{41.62}.$$

$$\begin{aligned} \log 63.45 &= 1.8024 \\ \log \sin 27^\circ 15' &= \overline{1.6607} \\ &\quad \overline{1.4631} \\ \log 41.62 &= \overline{1.6193} \\ \log \sin C &= \overline{1.8438}; \end{aligned}$$

whence $C = 44^\circ 16'$, or $135^\circ 44'$, both values being admissible since $b < c$.

$$33. \quad \sin A = \frac{a \sin B}{b} = \frac{17.28 \sin 55^\circ 13'}{23.97}.$$

$$\begin{aligned} \log 17.28 &= 1.2375 \\ \log \sin 55^\circ 13' &= \overline{1.9145} \\ &\quad \overline{1.1520} \\ \log 23.97 &= \overline{1.3797} \\ \log \sin A &= \overline{1.7723} \\ \therefore A &= 36^\circ 18' \\ \text{and } C &= 88^\circ 29'. \end{aligned}$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} = \frac{17.28 \sin 88^\circ 29'}{\sin 36^\circ 18'} \\ \log 17.28 &= 1.2375 \\ \log \sin 88^\circ 29' &= \overline{1.9999} \\ &\quad \overline{1.2374} \\ \log \sin 36^\circ 18' &= \overline{1.7723} \\ \log c &= \overline{1.4651} \\ \therefore c &= 29.18. \end{aligned}$$

34. $\sin B = \frac{b \sin A}{a} = \frac{141.3 \sin 40^\circ}{94.2}$.

$\log \sin B = 2.1501 + \bar{1}.8081 - 1.9741 = \bar{1}.9841$; whence $B = 74^\circ 36'$, or $105^\circ 24'$, since $a < b$. $\therefore C_1 = 65^\circ 24'$ and $C_2 = 34^\circ 36'$.

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{94.2 \sin 65^\circ 24'}{\sin 40^\circ}.$$

$$\log 94.2 = 1.9741$$

$$\log \sin 65^\circ 24' = \bar{1}.9587$$

$$\bar{1}.9328$$

$$\log \sin 40^\circ = \bar{1}.8081$$

$$\log c_1 = \bar{2}.1247$$

$$\therefore c_1 = 133.2.$$

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{94.2 \sin 34^\circ 36'}{\sin 40^\circ}.$$

$$\log 94.2 = 1.9741$$

$$\log \sin 34^\circ 36' = \bar{1}.7542$$

$$\bar{1}.7283$$

$$\log \sin 40^\circ = \bar{1}.8081$$

$$\log c_2 = \bar{1}.9202$$

$$\therefore c_2 = 83.22.$$

35. $\sin B = \frac{b \sin A}{a} = \frac{137 \sin 20^\circ 41'}{115}$.

$\therefore \log \sin B = 2.1367 + \bar{1}.5481 - 2.0607 = \bar{1}.6241$; whence $B = 24^\circ 53'$, or $155^\circ 7'$, since $a < b$. $\therefore C_1 = 134^\circ 26'$ and $C_2 = 4^\circ 12'$.

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{115 \sin 45^\circ 34'}{\sin 20^\circ 41'}.$$

$$\log 115 = 2.0607$$

$$\log \sin 45^\circ 34' = \bar{1}.8538$$

$$\bar{1}.9145$$

$$\log \sin 20^\circ 41' = \bar{1}.5481$$

$$\log c_1 = \bar{2}.3664$$

$$\therefore c_1 = 232.5.$$

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{115 \sin 4^\circ 12'}{\sin 20^\circ 41'}.$$

$$\log 115 = 2.0607$$

$$\log \sin 4^\circ 12' = \bar{2}.8647$$

$$\bar{2}.9254$$

$$\log \sin 20^\circ 41' = \bar{1}.5481$$

$$\log c_2 = \bar{1}.3773$$

$$\therefore c_2 = 23.84.$$

36. $\sin C = \frac{c \sin B}{b} = \frac{1665}{1325} \sin 52^\circ 19'$.

$$\log 1665 = 3.2214$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\bar{3}.1198$$

$\therefore C = 84^\circ$, or 96° , both values being admissible since $b < c$.

$$\log 1325 = 3.1222$$

$$\log \sin C = \bar{1}.9976$$

Now, with diagram of page 132, we have $\angle BAC_2 = 31^\circ 41'$;

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31^\circ 41'}{\sin 52^\circ 19'}.$$

$$\log 1325 = 3.1222$$

$$\log \sin 31^\circ 41' = \bar{1}.7203$$

$$\bar{2}.8425$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\log a = \bar{2}.9441; \text{ whence } a = 879.2.$$

37. $\sin A = \frac{a \sin C}{c} = \frac{324.7 \sin 35^\circ}{421.7}.$

$\log 324.7 = 2.5114$	$b = \frac{c \sin B}{\sin C} = \frac{421.7 \sin 61^\circ 12'}{35^\circ}$
$\log \sin 35^\circ = \underline{\overline{1.7586}}$	$\log 421.7 = 2.6250$
$\underline{\overline{2.2700}}$	$\log \sin 61^\circ 12' = \underline{\overline{1.9427}}$
$\log 421.7 = 2.6250$	$\log \sin 35^\circ = \underline{\overline{1.7586}}$
$\log \sin A = \underline{\overline{1.6450}}$	$\log b = \underline{\overline{2.8091}}$
$\therefore A = 26^\circ 12',$ and $B = 118^\circ 48'.$	$\therefore b = 644.3.$

38. $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9.99}{53.91} \cot 17^\circ 30'.$

$\log 9.99 = .9996$	
$\log \cot 17^\circ 30' = \underline{\overline{.5013}}$	
$\underline{\overline{1.5009}}$	$\frac{A+B}{2} = 72^\circ 30',$
$\log 53.91 = \underline{\overline{1.7317}}$	
$\log \tan \frac{A-B}{2} = \underline{\overline{1.7692}};$	whence $\frac{A-B}{2} = 30^\circ 27'.$
	$\therefore A = 102^\circ 57',$ and $B = 42^\circ 3'.$

39. $\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{549 \times 291}{1000 \times 1258}}.$

$\log 549 = 2.7396$	$\log 1000 = 3$
$\log 291 = 2.4639$	$\log 1258 = 3.0997$
$\underline{\overline{5.2035}}$	$\underline{\overline{6.0997}}$
$\underline{\overline{6.0997}}$	
$2) \underline{\overline{1.1038}}$	

$\log \sin \frac{B}{2} = \underline{\overline{1.5519}};$ whence $\frac{B}{2} = 20^\circ 52'.$

$\therefore B = 41^\circ 44'.$

40. $\sin B = \frac{b \sin C}{c} = \frac{17 \sin 43^\circ 12'}{12}.$

$\log 17 = 1.2304$	
$\log \sin 43^\circ 12' = \underline{\overline{1.8354}}$	$\therefore B = 75^\circ 51',$ or $104^\circ 9',$
$\underline{\overline{1.0658}}$	both values being admissible
$\log 12 = 1.0792$	since $c < b.$
$\log \sin B = \underline{\overline{1.9866}}$	
	$\therefore A = 60^\circ 57',$ or $32^\circ 39'.$

$$41. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}.$$

$$\begin{array}{rcl} \log 326 & = & 2.5132 \\ \log 199 & = & 2.2989 \\ & & \left| \begin{array}{l} 4.8121 \\ 5.6436 \\ \hline 2) \overline{1.1685} \end{array} \right. \\ & & \log 976 = 2.9894 \\ & & \log 451 = 2.6542 \\ & & \hline & & 5.6436 \end{array}$$

$$\log \tan \frac{A}{2} = \bar{1.5843}; \text{ whence } \frac{A}{2} = 21^\circ$$

$\therefore A = 42^\circ.$

$$\text{Again, } \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}}.$$

$$\begin{array}{rcl} \log 451 & = & 2.6542 \\ \log 199 & = & 2.2989 \\ & & \left| \begin{array}{l} 4.9531 \\ 5.5026 \\ \hline 2) \overline{1.4505} \end{array} \right. \\ & & \log 976 = 2.9894 \\ & & \log 326 = 2.5132 \\ & & \hline & & 5.5026 \end{array}$$

$$\log \tan \frac{B}{2} = \bar{1.7253}; \text{ whence } \frac{B}{2} = 27^\circ 59'.$$

$$\therefore B = 55^\circ 58', \text{ and } C = 82^\circ 2'.$$

$$42. \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2.5 \times 1.5}{5 \times 6}} = \sqrt{\frac{1}{8}}.$$

$$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2 = \bar{1.5485},$$

$$\therefore \frac{A}{2} = 20^\circ 42', \text{ and } A = 41^\circ 24'.$$

$$43. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11.29}{38.95} \cot 23^\circ 37'.5.$$

$$\log \cot 23^\circ 36' = .3596$$

$$\text{subtract for } 1.5 = \frac{5}{.3591} \text{ (the mean of diff. 3 and 7)}$$

$$\log 11.29 = 1.0527$$

$$\bar{1.4118}$$

$$\frac{B+C}{2} = 66^\circ 22'.5,$$

$$\log 38.95 = 1.5905$$

$$\log \tan \frac{B-C}{2} = \bar{1.8213}; \quad \text{whence } \frac{B-C}{2} = 33^\circ 32'.$$

$$\therefore B = 99^\circ 54'.5, \text{ and } C = 32^\circ 50'.5.$$

$$\text{Again, } a = \frac{b \sin A}{\sin B} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 99^\circ 54' \cdot 5} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 80^\circ 5' \cdot 5}.$$

$$\log 25 \cdot 12 = 1 \cdot 4000$$

$$\log \sin 47^\circ 15' = \overline{1} \cdot 8658$$

$$\quad \quad \quad \overline{1} \cdot 2658$$

$$\log \sin 80^\circ 5' \cdot 5 = \overline{1} \cdot 9935$$

$$\log a = \overline{1} \cdot 2723; \text{ whence } a = 18 \cdot 72.$$

$$44. \quad \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\log 4367 = 3 \cdot 6402$$

$$\log \cot 15^\circ 45' = \overline{1} \cdot 5498$$

$$\quad \quad \quad \overline{4} \cdot 1900$$

$$\log 4667 = \overline{3} \cdot 6691$$

$$\frac{C+B}{2} = 74^\circ 15',$$

$$\log \tan \frac{C-B}{2} = \overline{1} \cdot 5209;$$

$$\text{whence } \frac{C-B}{2} = 73^\circ 14'.$$

$$\therefore C = 147^\circ 29', \text{ and } B = 1^\circ 1'.$$

$$\text{Again, } a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 147^\circ 29'} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31'}.$$

$$\log 4517 = 3 \cdot 6549$$

$$\log \sin 31^\circ 30' = \overline{1} \cdot 7181$$

$$\quad \quad \quad \overline{3} \cdot 3730$$

$$\log \sin 32^\circ 31' = \overline{1} \cdot 7304$$

$$\log a = \overline{3} \cdot 6426; \text{ whence } a = 4391.$$

$$45. \quad \sin C = \frac{c \sin A}{a} = \frac{435 \cdot 6 \sin 36^\circ 18'}{321 \cdot 7}.$$

$$\log 435 \cdot 6 = 2 \cdot 6391$$

$$\log 36^\circ 18' = \overline{1} \cdot 7723$$

$$\quad \quad \quad \overline{2} \cdot 4114$$

$$\log 321 \cdot 7 = \overline{2} \cdot 5074$$

$$\log \sin C = \overline{1} \cdot 9040;$$

whence $C = 53^\circ 17'$, or $126^\circ 43'$, both values being admissible since $a < c$.

46. For the first part of the example, see Art. 197.

$$\tan \theta = \frac{2\sqrt{17 \cdot 32 \times 13 \cdot 47}}{3 \cdot 85} \sin 23^\circ 36'.$$

$$\log 17 \cdot 32 = 1 \cdot 2385$$

$$\log 13 \cdot 47 = 1 \cdot 1294$$

$$2) \overline{2} \cdot 3679$$

$$\quad \quad \quad \overline{1} \cdot 1840$$

$$\log 2 = \overline{1} \cdot 3010$$

$$\log \sin 23^\circ 36' = \overline{1} \cdot 6024$$

$$\quad \quad \quad \overline{1} \cdot 0874;$$

$$1 \cdot 0874$$

$$\log 3 \cdot 85 = \overline{1} \cdot 5855$$

$$\log \tan \theta = \overline{1} \cdot 5019$$

whence $\theta = 72^\circ 31'$, approx.

Again,

$$c = \frac{a - b}{\cos \theta} = \frac{3.85}{\cos 72^\circ 31'}.$$

$$\begin{aligned}\log 3.85 &= .5855 \\ \log \cos 72^\circ 31' &= \overline{1.4777} \\ \log c &= 1.1078; \text{ whence } c = 12.81.\end{aligned}$$

47. See Art. 195.

$$\tan \phi = \frac{44.1}{10.5} \tan 22^\circ 36'.$$

$$\begin{aligned}\log 44.1 &= 1.6444 \\ \log \tan 22^\circ 36' &= \overline{1.6194} \\ &\quad \overline{1.2638} \\ \log 10.5 &= \overline{1.0212} \\ \log \tan \phi &= \overline{.2426}; \text{ whence } \phi = 60^\circ 14', \text{ approx.}\end{aligned}$$

$$\text{Again, } c = \frac{(a - b) \cos \frac{C}{2}}{\cos \phi} = \frac{10.5 \cos 22^\circ 36'}{\cos 60^\circ 14'}.$$

$$\begin{aligned}\log 10.5 &= 1.0212 \\ \log \cos 22^\circ 36' &= \overline{1.9653} \\ &\quad \overline{.9865} \\ \log \cos 60^\circ 14' &= \overline{1.6959} \\ \log c &= 1.2906; \text{ whence } c = 19.53.\end{aligned}$$

EXAMPLES. XVII. a. PAGE 185.

1. See figure on page 184.

Let PC represent the cliff, and A and B the two objects. Then $PC = 200$ ft.; $\angle PAC = 30^\circ$, $\angle PBC = 45^\circ$.

$$AB = \frac{BP \sin APB}{\sin PAB} = \frac{BP \sin 15^\circ}{\sin 30^\circ};$$

and

$$BP = \frac{PC}{\sin PBC} = \frac{200}{\sin 45^\circ};$$

$$\therefore AB = \frac{200 \sin 15^\circ}{\sin 45^\circ \sin 30^\circ} = 200 (\sqrt{3} - 1) = 146.4 \text{ ft.}$$

2. See figure on page 185.

Let P represent the mountain top, and A, B the two positions of the observer.

Then $\angle PAC = 15^\circ$, $\angle PBC = 75^\circ$, $AB = 1$ mile.

Let x be the height of mountain in feet;

$$\text{then } x = PB \sin 75^\circ; \text{ and } PB = \frac{AB \sin 15^\circ}{\sin 60^\circ};$$

$$\begin{aligned}\therefore x &= \frac{1760 \cdot 3 \cdot \sin 15^\circ \sin 75^\circ}{\sin 60^\circ} = \frac{880 \cdot 3 \cdot (\cos 60^\circ - \cos 90^\circ)}{\sin 60^\circ} \\ &= \frac{880 \cdot 3}{\sqrt{3}} = 880\sqrt{3} = 1524 \text{ ft.}\end{aligned}$$

3. Let A, B be the position of the two forts, P the first position of the ship, and Q its position after moving 4 miles towards A ;

then $PQ = 4$ miles, $\angle QPB = 30^\circ$, $\angle AQB = 48^\circ$; $\therefore \angle QBP = 18^\circ$.

$$\therefore QB = \frac{QP \sin 30^\circ}{\sin 18^\circ} = \frac{8}{\sqrt{5}-1} = 2(\sqrt{5}+1) = 6.472 \text{ miles.}$$

4. See figure on page 184.

Let PC represent the tower and A, B the two objects; then

$$PC = h, \angle PAC = 45^\circ - A, \angle PBC = 45^\circ + A, \angle APB = 2A;$$

$$\therefore AB = \frac{PB \sin 2A}{\sin(45^\circ - A)}, \text{ and } PB = \frac{h}{\sin(45^\circ + A)};$$

$$\therefore AB = \frac{2h \sin 2A}{2 \sin(45^\circ - A) \sin(45^\circ + A)} = \frac{2h \sin 2A}{\cos 2A - \cos 90^\circ} = 2h \tan 2A.$$

5. In the figure on page 184, take D in AB , so that $CD = CP$; then

$$AD = a \text{ feet}, DB = b \text{ feet}, PC = x \text{ feet, suppose.}$$

$$\text{Since } \angle CDP = 45^\circ = \angle DPC, \therefore DC = PC = x.$$

$$\text{Also } AC = x + a, BC = x - b, \angle BPC = A.$$

$$\text{Now from } \triangle PAC, \tan A = \frac{x}{x+a} = \frac{x-b}{x}, \text{ in } \triangle BPC.$$

$$\therefore x^2 = (x+a)(x-b); \text{ whence } x = \frac{ab}{a-b}.$$

6. Let P , Q be the two positions of the observer.

Then from the figure, since

$$\angle AQP = 90^\circ, \angle QPA = 45^\circ;$$

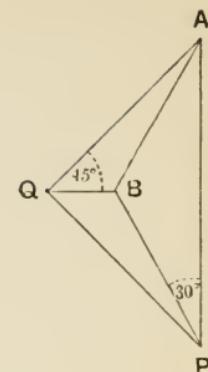
$$\therefore QA = QP = 1 \text{ mile.}$$

And Δ 's ABQ , PBQ are equal in all respects

(Euc. i. 4);

$$\therefore \angle ABQ = \angle PBQ = 120^\circ;$$

$$\therefore AB = \frac{AQ \sin 45^\circ}{\sin 120^\circ} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{3}\sqrt{6} = .816 \text{ miles.}$$



7. Let A be the base, and B the top of the tower, and let C be the point of observation 40 feet up the hill.

Then $AC = 40$ ft., $\angle BAC = 90^\circ - 9^\circ = 81^\circ$, $\angle BCA = 54^\circ$; $\therefore \angle ABC = 45^\circ$;

$$\therefore AB = \frac{AC \sin 54^\circ}{\sin 45^\circ} = 10\sqrt{2}(\sqrt{5} + 1) = 45.76 \text{ feet.}$$

8. See figure on page 186.

Let PA represent the tower, and PB the flagstaff.

Then $PB = c$ feet, $\angle PCA = \alpha$, $\angle PCB = \beta$, and we have

$$x = CP \sin \alpha = \frac{c \cos(\alpha + \beta) \sin \alpha}{\sin \beta}.$$

9. See figure on page 186.

Let BP represent the flagstaff, and PA the wall, C the point of observation.

Let $\angle BCP = \alpha$, $\angle PCA = \theta$, $CA = a$,

Then $\tan \alpha = .5$, $BP = 20$ ft., $PA = 10$ ft.

$$\text{Now } \tan(\theta + \alpha) = \frac{30}{a}, \quad \tan \theta = \frac{10}{a};$$

$$\therefore 3 \tan \theta = \tan(\theta + \alpha) = \frac{2 \tan \theta + 1}{2 - \tan \theta};$$

$$\therefore 3 \tan^2 \theta - 4 \tan \theta + 1 = 0; \text{ whence } \tan \theta = 1 \text{ or } \frac{1}{3}.$$

10. See figure on page 186.

Let BP represent the statue, PA the tower, and C the point of observation.

Let $\angle PCA = \alpha$, $\angle BCP = \beta$, $BP = x$ feet.

Then $PA = 25$ ft., $CA = 60$ ft.;

$$\therefore \tan \alpha = \frac{25}{60} = \frac{5}{12}, \text{ and } \tan \beta = \frac{125}{8} = \frac{1}{8}.$$

$$\text{Now } x + 25 = 60 \tan(\alpha + \beta) = 60 \left(\frac{\frac{1}{8} + \frac{5}{12}}{1 - \frac{5}{96}} \right) = 34\frac{2}{7};$$

$$\therefore \text{height of statue} = 9\frac{2}{7} \text{ feet.}$$

11. See figure and example on page 187.

Here we have $BC = 9$ ft., $BD = 289$ ft., $BE = 324$ ft.;

$$\therefore \tan(\alpha + \theta) = \frac{324}{x}; \quad \tan \alpha = \frac{289}{x}; \quad \tan \theta = \frac{9}{x}.$$

$$\text{But } \tan(\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta};$$

$$\therefore \frac{324}{x} = \frac{\frac{289}{x} + \frac{9}{x}}{1 - \frac{289}{x} \cdot \frac{9}{x}} = \frac{298}{x} \cdot \frac{x^2}{x^2 - 289 \times 9};$$

$$\therefore 324x^2 - 324 \times 289 \times 9 = 298x^2;$$

$$\therefore 26x^2 = 324 \times 289 \times 9 = 18^2 \times 17^2 \times 3^2;$$

$$\therefore x^2 = \frac{18^2 \times 51^2}{26} = 18^2 \times \left(\frac{50}{5}\right)^2 \text{ nearly;}$$

thus $x = 180$ ft. nearly.

12. See figure on page 187.

Let BD represent the column, DE the statue, BC the man standing by the column, A the point on the opposite bank of the river.

Then $BC = 6$ ft., $BD = 192$ ft., $BE = 216$ ft.

Let $AB = x$ ft., $\angle EAD = \angle CAB = \theta$, $\angle DAB = \alpha$.

$$\text{Then } \tan(\alpha + \theta) = \frac{216}{x}; \quad \tan \alpha = \frac{192}{x}; \quad \tan \theta = \frac{6}{x};$$

$$\therefore \frac{216}{x} = \frac{\frac{192}{x} + \frac{6}{x}}{1 - \frac{6}{x} \times \frac{192}{x}} = \frac{198}{x} \cdot \frac{x^2}{x^2 - 6 \times 192}.$$

From this equation we obtain $x = 48\sqrt{6}$;

\therefore breadth of river $= 48\sqrt{6} = 117.6$ feet nearly.

13. We have at once from a figure,

$$\tan \alpha = \frac{a}{x}, \quad \tan \beta = \frac{b}{x}, \quad \tan \gamma = \frac{c}{x}.$$

Now

$$\alpha + \beta + \gamma = 180^\circ;$$

$$\therefore \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma;$$

$$\therefore \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{abc}{x^3};$$

that is,

$$(a+b+c)x^2 = abc.$$

14. See figure on page 188.

Let P be the top of the hill, A and B the points of observation; then

$$\angle PAC = 45^\circ, \angle BAC = 15^\circ, \angle PDC = 75^\circ;$$

$$\therefore \angle BPA = 75^\circ - 45^\circ = 30^\circ = \angle PAB;$$

$$\therefore PA = 2AB \cos 30^\circ = 500\sqrt{3} \text{ yards};$$

$$\begin{aligned} \therefore \text{height of hill} &= PC = PA \sin 45^\circ = 250\sqrt{3}/6 \text{ yards} \\ &= 750\sqrt{6} \text{ feet.} \end{aligned}$$

15. We have $\angle CBA = 30^\circ, \angle BCA = 135^\circ; \therefore \angle BAC = 15^\circ;$

$$\begin{aligned} \therefore AB &= \frac{BC \sin 135^\circ}{\sin 15^\circ} = 1760 \times 3 \times \frac{2}{\sqrt{3}-1} \text{ ft.} \\ &= 1760 \times 3 (\sqrt{3}+1) \text{ feet;} \end{aligned}$$

$$\therefore \text{height of mountain} = AB \sin 60^\circ = 880 \times 3 (3+\sqrt{3}) = 12492 \text{ ft.}$$

16. See figure on page 188.

Let A, B be the two points of observation and P the top of the hill; then in the figure $\angle PAC = \alpha, \angle PDC = \gamma, AB = c \text{ ft.};$

$$\therefore \angle APB = \gamma - \alpha, \angle ABP = \pi - (\gamma - \beta),$$

$$\text{and } AP = \frac{AB \sin (\gamma - \beta)}{\sin (\gamma - \alpha)};$$

$$\begin{aligned} \therefore \text{height of hill} &= AP \sin \alpha \\ &= c \sin \alpha \sin (\gamma - \beta) \operatorname{cosec} (\gamma - \alpha) \text{ feet.} \end{aligned}$$

17. In the figure let P be the top of the mountain, and A, B the two points of observation.

Then $AE = EB = 800 \text{ ft.};$

$$\angle BAE = 15^\circ, \angle BED = 30^\circ.$$

$$\text{Also } \angle PDC = 75^\circ, \angle PAC = 60^\circ;$$

$$\therefore \angle APD = 15^\circ.$$

From $\triangle ABE$ we have

$$AB = 2AE \cos 15^\circ = 1600 \cos 15^\circ \text{ ft.};$$

$$\text{and from } \triangle APB, AP = \frac{AB \sin 120^\circ}{\sin 15^\circ}$$

$$= 800\sqrt{3} \cot 15^\circ = 800\sqrt{3} (2+\sqrt{3}) \text{ ft.};$$

$$\therefore \text{height of mountain} = AP \sin 60^\circ$$

$$= 400 \times 3 (2+\sqrt{3}) = 4478 \text{ ft.}$$

approximately.



EXAMPLES. XVII. b. PAGE 190.

1. In the figure of page 186 let $BP=25$ ft., $PA=15$ ft., $CA=x$ feet.

Then $\frac{BC}{x} = \frac{25}{15} = \frac{5}{3}$ [Euc. vi. 3];

$$\therefore \frac{x^2 + 40^2}{x^2} = \frac{25}{9}; \text{ whence } x=30;$$

thus the width of the road is 30 ft.

2. Let C be the position of the observer, B the top of the statue, A the foot of the column.

Then if $CB=x$ feet, we have

$$\frac{x}{CA} = \frac{a}{3a} = \frac{1}{3} \quad [\text{Euc. vi. 3}],$$

that is, $\frac{x}{\sqrt{x^2+16a^2}} = \frac{1}{3}$; whence $x=a\sqrt{2}$.

3. See figure on page 189.

Let BL be the flagstaff, LA the tower, C the observer; then $BL=a$, $LA=b$, $DA=CE=d$, $EA=CD=h$; also $BC^2=CE^2+EB^2=d^2+(a+b-h)^2$,

$$CA^2=CE^2+EA^2=d^2+h^2.$$

Now $\frac{BC}{CA} = \frac{BL}{LA}; \therefore \frac{(a+b-h)^2+d^2}{h^2+d^2} = \frac{a^2}{b^2};$

$$\therefore \frac{(a+b)^2 - 2h(a+b)}{h^2+d^2} = \frac{a^2-b^2}{b^2};$$

$$\frac{a+b-2h}{h^2+d^2} = \frac{a-b}{b^2};$$

or whence $(a-b)d^2 = (a+b)b^2 - 2b^2h - (a-b)h^2$.

4. See figure on page 190.

From O , the centre of the circle, draw OL , OM perpendicular to AB and DC respectively; then L , M bisect AB , DC . Let $DC=2x$ feet.

Then $\angle \beta = \frac{1}{2} \angle DOC$ at centre $= \angle COM$.

Now $CD=2x=2CM=2OM \tan \beta$
 $=2EL \tan \beta=(a+b) \tan \beta,$

since $2EL=EB+EA=a+b$.

5. With the same figure and notation as in the last Example, we have $ED=AB=20$ ft., and $\beta=45^\circ$.

$$\therefore 2x=(EB+EA) \tan 45^\circ=2EB+20;$$

$$\therefore x=EB+10.$$

Again $ED \cdot EC = EB \cdot EA = EB(EB + 20)$.

$$\therefore 20(20 + 2x) = (x - 10)(x + 10);$$

whence $x = 50$.

Thus the height of the column is 100 ft.

6. Take the figure on page 190, interchanging the letters A and B . Then we have

$$\angle BDA = \angle BCA = \alpha - \beta; \quad \angle ABD = \angle ACE = 90^\circ - \alpha;$$

$$\therefore \angle DBC = \beta - \angle ABD = \alpha + \beta - 90^\circ.$$

Now from $\triangle CBD$,

$$CD = \frac{BD \sin DBC}{\sin BCE} = \frac{BD \sin (\alpha + \beta - 90^\circ)}{\sin BCE};$$

$$BD = \frac{AB \sin BAD}{\sin BDA} = \frac{a \sin BCE}{\sin (\alpha - \beta)};$$

$$\therefore CD = a \sin (\alpha + \beta - 90^\circ) \operatorname{cosec} (\alpha - \beta).$$

7. See figure on page 191.

Let CB be the pillar, BA the pedestal, E the point where the pillar subtends its maximum angle 30° .

Then using the same construction as in Ex. III. page 191, we have

$$\alpha = 30^\circ, \quad EA = 60 \text{ ft.}$$

Now $\angle AEB = \angle ECB = \frac{1}{2} \angle EDB = \frac{1}{2}(90^\circ - 30^\circ) = 30^\circ$.

$$CB = 2CF = 2DF \tan 30^\circ = 2 \times 60 \times \frac{1}{\sqrt{3}} = 40\sqrt{3} \text{ ft.}$$

8. Let O, P be the two positions of the observer; let

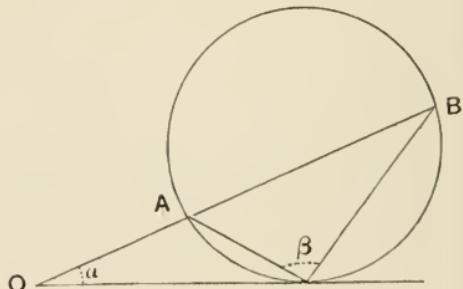
$$\angle APO = \theta.$$

Then $\angle ABP$, in alternate segment, $= \theta$.

$$\text{Now } AB = \frac{AP \sin \beta}{\sin \theta}$$

$$= \frac{c \sin \alpha \sin \beta}{\sin \theta \sin (\alpha + \theta)}$$

$$= \frac{2c \sin \alpha \sin \beta}{\cos \alpha - \cos (\alpha + 2\theta)}.$$



But, from $\triangle OPB$, $\alpha + 2\theta + \beta = 180^\circ$;

$$\therefore AB = 2c \sin \alpha \sin \beta / (\cos \alpha + \cos \beta).$$

9. Let OA be the tower, AB the flagstaff, P, Q the points at which the flagstaff subtends equal angles, R the point at which it subtends the greatest possible angle; then since $\angle APB = \angle AQB$;

$$\therefore B, A, P, Q \text{ are concyclic and } OP \cdot OQ = OA \cdot OB.$$

Again since ARB is the greatest angle subtended at a point in OQ by the str. line AB ; \therefore a circle can be drawn to pass through A, B and touch OQ at R ;

$$\therefore OA \cdot OB = OR^2;$$

$$\therefore OP \cdot OQ = OR^2;$$

that is OP, OR, OQ are in geometrical progression.

10. Here $ABCD$ is a cyclic quadrilateral;

$$\therefore \frac{AB}{\sin ADB} = \frac{BD}{\sin BAD} = \frac{BD}{\sin BCD} = \frac{DC}{\sin CBD};$$

$$\therefore AB \sin CBD = CD \sin ADB.$$

11. Since $ABED$ is a cyclic quadrilateral, we have

$$\angle ADC = \angle EBC = \gamma, \text{ and } \angle BDC = \beta.$$

Also $\angle BDA = \gamma - \beta$, and $\angle ACE = \pi - (\alpha + \beta + \gamma)$.

$$\therefore BC = \frac{BD \sin \beta}{\sin(\alpha + \beta + \gamma)} = \frac{AB \sin(\alpha + \beta) \sin \beta}{\sin(\gamma - \beta) \sin(\alpha + \beta + \gamma)}.$$

12. Here the points P, Q, R, S, A are concyclic, and

$$\angle RAS = \angle PAQ,$$

since AR, AS are perpendicular to AP, AQ ;

$$\therefore PQ = RS = \sqrt{400 + 100 - 2 \times 200 \cos 30^\circ}$$

$$= \sqrt{500 - 200\sqrt{3}} = 12.4 \text{ ft. nearly.}$$

13. Let A, B be the two beacons, P, Q the positions of the ship at the end of 3 min. and 21 min. respectively. Let $\angle ABP = \alpha$, $\angle PBQ = \theta$. Then it is easily seen that

$$\angle OAP = 90^\circ + \alpha,$$

$$\angle OAQ = 90^\circ + \alpha + \theta.$$

Also from the $\triangle OBQ$ we have

$$\alpha + \theta + 90^\circ + \alpha = 135^\circ,$$

so that $\alpha + \theta = 45^\circ - \alpha$.

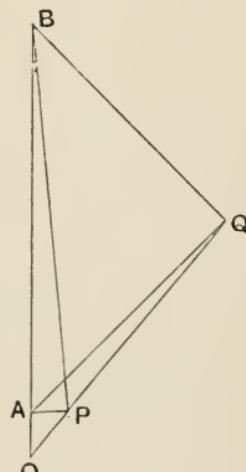
If $PO = x, OQ = 7x$;

$$\begin{aligned} \text{hence } x &= \frac{AP}{\sin 45^\circ} \sin(90^\circ + \alpha) \\ &= AB \sqrt{2} \cdot \sin \alpha \cos \alpha \\ &= \frac{5}{\sqrt{2}} \sin 2\alpha \quad \dots \dots \dots (1). \end{aligned}$$

$$\text{Again } 7x = \frac{AQ}{\sin 45^\circ} \sin(90^\circ + \alpha + \theta)$$

$$= AB \sqrt{2} \sin(\alpha + \theta) \cos(\alpha + \theta)$$

$$= \frac{5}{\sqrt{2}} \sin 2(\alpha + \theta) = \frac{5}{\sqrt{2}} \sin(90 - 2\alpha) = \frac{5}{\sqrt{2}} \cos 2\alpha. \quad \dots \dots \dots (2).$$



Squaring and adding (1) and (2), we have

$$50x^2 = \frac{25}{2}; \quad \therefore x = \frac{1}{2} \text{ mile.}$$

\therefore the ship sails a mile in 6 min., or at the rate of 10 miles an hour.

Again if y miles be the distance from O at which the beacons subtend the greatest angle, we have

$$y^2 = OP \cdot OQ = \frac{7}{4}; \quad \therefore y = \frac{\sqrt{7}}{2}.$$

And the ship will travel this distance in $\frac{\sqrt{7}}{2} \times \frac{60}{10}$, or $3\sqrt{7}$ minutes.

14. Let AB be the flag-staff, BC the tower, D and E the first and second positions of the observer respectively.

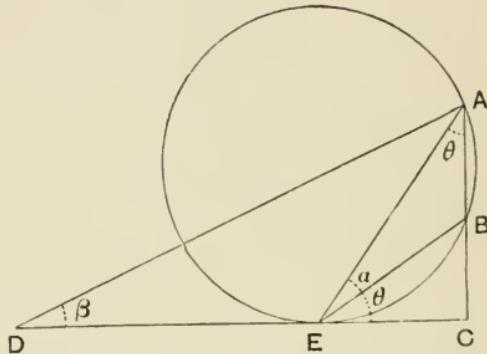
Then since AB subtends the maximum angle at E , the circle round ABE touches DC at E , so that

$\angle BEC = \angle EAB = \theta$, suppose.

$$\therefore a + \theta = \angle EBC = 90^\circ - \theta;$$

$$\therefore 2\theta = 90^\circ - a;$$

$$\begin{aligned} \text{also } \angle EAD &= a + \theta - \beta \\ &= 90^\circ - \theta - \beta. \end{aligned}$$



$$\text{Now } AB = \frac{AE \sin \alpha}{\sin(\theta + \alpha)}, \quad AE = \frac{ED \sin \beta}{\sin(90^\circ - \theta - \beta)} = \frac{a \sin \beta}{\cos(\theta + \beta)};$$

$$\begin{aligned} \therefore AB &= \frac{a \sin \alpha \sin \beta}{\sin(\theta + \alpha) \cos(\theta + \beta)} = \frac{2a \sin \alpha \sin \beta}{\sin(2\theta + a + \beta) + \sin(a - \beta)} \\ &= \frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin(a - \beta)}. \end{aligned}$$

EXAMPLES. XVII. c. PAGE 195.

1. Let A be the top of the hill and B its projection on the horizontal plane through P , Q .

Let

$$AB = x \text{ yards};$$

then

$$BP = BA = x,$$

$$BQ = BA \cot 30^\circ = x\sqrt{3};$$

$$\therefore 3x^2 = x^2 + 500^2;$$

$$\therefore x = 250\sqrt{2};$$

that is, height of the hill = $250\sqrt{2}$ yards = 1060.5 feet.

2. Let P be the top, and Q the bottom of the spire;

then $AQ = 250 \cot 60^\circ = \frac{250}{\sqrt{3}}$ feet,

$$BQ = 250 \cot 30^\circ = 250\sqrt{3} \text{ feet};$$

$$\therefore AB^2 = BQ^2 - AQ^2 = 250^2 \left(3 - \frac{1}{3} \right);$$

$$\therefore AB = 250 \cdot \frac{2\sqrt{2}}{\sqrt{3}} = \frac{500\sqrt{6}}{3} \text{ feet.}$$

3. Let P be the top and Q the bottom of the tower;

then $\angle PAQ = 60^\circ$; $\therefore QA = 360 \cot 60^\circ = \frac{360}{\sqrt{3}}$ feet and $QB = QP = 360$ feet;

$$\therefore \text{breadth of river} = AB = \sqrt{BQ^2 - QA^2} = 360 \sqrt{1 - \frac{1}{3}} = 120\sqrt{6} \text{ feet.}$$

4. See figure on page 194.

Let CD be the steeple, then $\angle CAD = 45^\circ$; $\therefore CA = CD = x$;

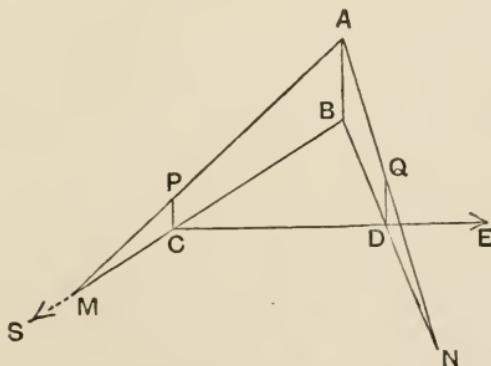
and $\angle CBD = 15^\circ$; $\therefore CB = x \cot 15^\circ = x(2 + \sqrt{3})$;

$$\therefore x^2(7 + 4\sqrt{3}) = x^2 + 4a^2;$$

$$\therefore x^2 = \frac{2a^2}{2\sqrt{3} + 3} = \frac{a^2(4 - 2\sqrt{3})}{\sqrt{3}};$$

$$\therefore \text{height of steeple} = \frac{a(\sqrt{3} - 1)}{\sqrt[4]{3}} = a(3^{\frac{1}{4}} - 3^{-\frac{1}{4}}).$$

5. Let AB be the lighthouse, and let CP, DQ represent the two positions of the observer, M, N the extremities of his shadow at each place.



Let $AB = x$ feet, then $x : BM = PC : CM = 6 : 24$;

$$\therefore BC = BM - CM = 4x - 24.$$

Also

$$x : BN = QD : DN = 6 : 30;$$

$$\therefore BD = BN - DN = 5x - 30;$$

$$\therefore 25(x-6)^2 = 300^2 + 16(x-6)^2;$$

$$\therefore 9(x-6)^2 = 300^2;$$

whence

$$x = 106 \text{ or } -94;$$

\therefore the light is 106 feet from the ground.

6. Let A be the balloon, and C, B, D the points of observation at which the angles of elevation of the balloon are $60^\circ, 30^\circ, 45^\circ$ respectively.

Let x yards be the height of the balloon;

$$\text{then } AB = x \operatorname{cosec} 30^\circ = 2x, AC = x \operatorname{cosec} 60^\circ = \frac{2x}{\sqrt{3}}, AD = x \operatorname{cosec} 45^\circ = x\sqrt{2}.$$

Now

$$2AD^2 + 2BD^2 = AB^2 + AC^2;$$

$$\therefore 4x^2 + 2 \times 880^2 = 4x^2 + \frac{4x^2}{3};$$

whence

$$x = 880 \cdot \sqrt{\frac{3}{2}} = 440\sqrt{6};$$

$$\therefore \text{height of balloon} = 440\sqrt{6} \text{ yards.}$$

7. Let A be the top of the mountain, BC the base of length $2a$, D its middle point.

Let x be the height of the mountain;

then

$$AB = AC = x \operatorname{cosec} \theta,$$

$$AD = x \operatorname{cosec} \phi.$$

And we have

$$AD^2 + BD^2 = AB^2;$$

$$\therefore x^2 \operatorname{cosec}^2 \phi + a^2 = x^2 \operatorname{cosec}^2 \theta;$$

$$\therefore x^2 = \frac{a^2}{\operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \phi} = \frac{a^2 \sin^2 \theta \cos^2 \theta}{\sin^2 \phi - \sin^2 \theta},$$

$$\therefore x = \frac{a \sin \theta \cos \theta}{\sqrt{\sin(\phi + \theta) \sin(\phi - \theta)}};$$

$$\therefore \text{height of mountain is } a \sin \theta \cos \theta \sqrt{\operatorname{cosec}(\phi + \theta) \operatorname{cosec}(\phi - \theta)}.$$

8. Let AB, CD be the two vertical poles, E the point in the line BD joining their feet at which each subtends an angle α , and F any point in the horizontal plane such that $\angle DFB$ is a right angle;

then $\angle CED = \angle AEB = \alpha, \angle AFB = \beta, \angle CED = \gamma$;

and $EB = AB \cot AEB = a \cot \alpha, ED = b \cot \alpha$;

$$\therefore BD = (a+b) \cot \alpha;$$

$$BF = AB \cot AFB = a \cot \beta, DF = b \cot \gamma;$$

and

$$BD^2 = BF^2 + DF^2;$$

$$\therefore (a+b)^2 \cot^2 \alpha = a^2 \cot^2 \beta + b^2 \cot^2 \gamma.$$

9. Let A be the top of the hill, B its projection on the horizontal plane in which the road lies, C, D, E the three consecutive milestones whose angles of depression are observed.

Then $\angle ACB = \alpha$, $\angle ADB = \beta$, $\angle AEB = \gamma$,
and $ED = DC = 1760 \times 3$ feet.

Let x feet be the height of the hill;
then since $BC^2 + BE^2 = 2BD^2 + 2DE^2$;

$$\therefore x^2 \cot^2 \alpha + x^2 \cot^2 \gamma = 2x^2 \cot^2 \beta + 2 \times 1760^2 \times 3^2;$$

$$\therefore x = \frac{5280\sqrt{2}}{\sqrt{\cot^2 \alpha - 2 \cot^2 \beta + \cot^2 \gamma}}.$$

10. Let P, Q be the two positions of the observer.

Let x feet be the height of the chimneys;

then $AP = x \cot 60^\circ = \frac{x}{\sqrt{3}}$, $AQ = AB = x$;
 $\therefore x^2 = \frac{x^2}{3} + 80^2$;

whence $x = 40\sqrt{6}$; \therefore height of chimney $= 40\sqrt{6}/6$ feet.

Again $CQ = x \cot 30^\circ = x\sqrt{3}$;

$$\therefore CP = \sqrt{CQ^2 - PQ^2} = \sqrt{3 \times 40^2 \times 6 - 80^2} = 40\sqrt{14};$$

$$\therefore AC = 40\sqrt{14} + 40\sqrt{2} = 40(\sqrt{14} + \sqrt{2}) = 206 \text{ feet nearly.}$$

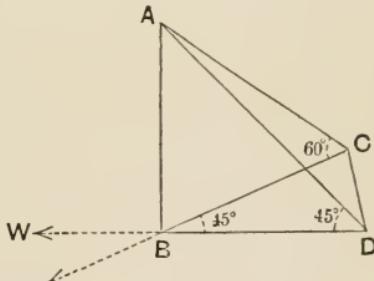
11. Let A be the balloon, C, D the positions of the two observers, B the point vertically below A in the horizontal plane of the observers.

Let $AB = x$ yds. = height of the balloon;
then $CD = 500$ yards,

$$BC = x \cot 60^\circ = \frac{x}{\sqrt{3}}, \quad BD = x;$$

and $CD^2 = BC^2 + BD^2 - 2BC \cdot BD \cos 45^\circ$;

$$\therefore 500^2 = \frac{x^2}{3} + x^2 - \frac{x^2\sqrt{6}}{3};$$



whence $x^2 = 50^2(120 + 30\sqrt{6})$;

$$\therefore \text{height of balloon} = 50\sqrt{120 + 30\sqrt{6}} = 696 \text{ yds. nearly.}$$

12. In the diagram on page 195, let AC be a line of greatest slope on the hill, and let CH be the railway.

Then $\angle ACB = \alpha$, $\angle BCG = \gamma$, $\angle HCG = \beta$, and $\angle CBG$ is a right angle.
Let $AB = HG = h$.

Then $BC = h \cot \alpha$, $CG = h \cot \beta$.

$$\therefore \cos \gamma = \frac{BC}{CG} = \cot \alpha \tan \beta.$$

EXAMPLES. XVII. d. PAGE 197.

1. Let C_1 , C_2 be the positions of the two churches, M the position of the man. Then MC_1C_2 is an isosceles triangle, and the vertical angle at $M=11^\circ 24'$.

If x miles is the height of the balloon, we have

$$x = \frac{1}{2} \cot 5^\circ 42'$$

$$= \frac{1}{2} \times 10.02$$

$$= 5.01.$$

2. Let DC be the pole, then from a figure we have

$$DC = BC \sin 10^\circ 45' = \frac{100 \sin 5^\circ 30' \times \sin 10^\circ 45'}{\sin 5^\circ 15'}.$$

$$\log 100 + \log \sin 5^\circ 30' = .9816$$

$$\begin{array}{r} \log \sin 10^\circ 45' = \bar{1}.2707 \\ \hline .2523 \end{array}$$

$$\log \sin 5^\circ 15' = \bar{2}.9612$$

$$\log DC = \overline{1.2911}$$

$$\therefore DC = 19.54 \text{ yards.}$$

Again,

$$BC = \frac{DC}{\tan 10^\circ 45'}.$$

$$\log DC = 1.2911$$

$$\log \tan 10^\circ 45' = \bar{1}.2785$$

$$\log BC = 2.0126$$

$$\therefore BC = 102.9 \text{ yards.}$$

3. If the required height is x feet, we have

$$x = \frac{500 \sin 26^\circ 34' \times \sin 12^\circ 32'}{\sin 14^\circ 2'}.$$

$$\begin{aligned} \log 500 &= 2.6990 \\ \log \sin 26^\circ 34' &= \bar{1}.6505 \\ \log \sin 12^\circ 32' &= \bar{1}.3364 \\ &\quad \overline{1.6859} \\ \log \sin 14^\circ 2' &= \bar{1}.3847 \\ \log x &= \overline{2.3012} \\ \therefore x &= 200.1. \end{aligned}$$

4. Let B be the position of the boat, FC the flagstaff, CD the cliff, then, if $CD=x$ feet, we have

$$\begin{aligned} x &= \frac{30 \sin 43^\circ 46'}{\sin 2^\circ 6'} \times \sin 44^\circ 8'. \\ \log 30 &= 1.4771 \\ \log \sin 43^\circ 46' &= \bar{1}.8399 \\ \log \sin 44^\circ 8' &= \bar{1}.8429 \\ &\quad \overline{1.1599} \\ \log \sin 2^\circ 6' &= \bar{2}.5640 \\ \log x &= \overline{2.5959} \\ \therefore \text{height} &= 394.4 \text{ ft.} \end{aligned}$$

Again,

$$\begin{aligned} BD &= \frac{x}{\tan 44^\circ 8'}. \\ \log x &= 2.5959 \\ \log \tan 44^\circ 8' &= \bar{1}.9869 \\ \log BD &= \overline{2.6090} \\ \therefore \text{distance} &= 406.4 \text{ ft.} \end{aligned}$$

5. With the figure on page 184, we have

$$\angle PBC = 10^\circ, \angle PAC = 5^\circ; PB = AB = 5280 \text{ ft.}$$

Now

$$PC = PB \sin 10^\circ = 5280 \sin 10^\circ.$$

$$\begin{aligned} \log 5280 &= 3.7226 \\ \log \sin 10^\circ &= \bar{1}.2397 \\ &\quad \overline{2.9623} \\ \therefore PC &= 916.8 \text{ ft.} \end{aligned}$$

Again,

$$\begin{aligned} BC &= BP \cos 10^\circ \\ &= (1 \times \cos 10^\circ) \text{ miles} \\ &= .9848 \text{ miles.} \end{aligned}$$

6. Let B be the point in the road which is vertically below the observer A . Let DC be the telegraph post, and let a horizontal line through A meet DC in E . Then $\angle DAE = 17^\circ 19'$, $\angle ACB = 8^\circ 36'$.

$$AE = BC = \frac{15}{\tan 8^\circ 36'}.$$

$$\log 15 = 1.1761$$

$$\log \tan 8^\circ 36' = \overline{1.1797}$$

$$\log AE = 1.9964$$

$$\therefore AE = 99.17 \text{ ft.}$$

Again,

$$DE = EA \tan 17^\circ 19'.$$

$$\log EA = 1.9964$$

$$\log \tan 17^\circ 19' = \overline{1.4938}$$

$$\log DE = \overline{1.4902}$$

$$\therefore DE = 30.91 \text{ ft.,}$$

$$\text{and } DC = 45.91 \text{ ft.}$$

7. Here we may take the third figure on page 131. Then $AC = 60$ miles, $CB_2 = CB_1 = 30$ miles, $\angle CAB_2 = 20^\circ 16'$.

$$\sin B = \frac{60}{30} \sin 20^\circ 16' = 2 \times .3464 = .6928;$$

$$\therefore B = 43^\circ 51', \text{ or } 136^\circ 9' \text{ (since } a < b).$$

$$\therefore \angle ACB_2 = 23^\circ 35', \quad \angle ACB_1 = 180^\circ - 64^\circ 7'.$$

Now $AB_2 = \frac{30 \sin 23^\circ 35'}{\sin 20^\circ 16'}.$

$$\log 30 = 1.4771$$

$$\log \sin 23^\circ 35' = \overline{1.6022}$$

$$\overline{1.0793}$$

$$\log \sin 20^\circ 16' = \overline{1.5396}$$

$$\log AB_2 = 1.5397$$

$$\therefore AB_2 = 34.65 \text{ miles.}$$

Again,

$$AB_1 = \frac{30 \sin 64^\circ 7'}{\sin 20^\circ 16'}.$$

$$\begin{array}{r} \log 30 = 1.4771 \\ \log \sin 64^\circ 7' = \overline{1.9541} \\ \hline 1.4312 \end{array}$$

$$\log \sin 20^\circ 16' = \overline{1.5396}$$

$$\begin{array}{r} \log AB_1 = 1.8916 \\ \therefore AB_1 = 77.91 \text{ miles.} \end{array}$$

Thus the train must travel at the rate of 11.55 miles or 25.97 miles per hour.

8. Let C be the doorstep, E the point of observation on the roof; then $CE = h$. Let AB be the spire and let ED drawn horizontally meet AB in D ; then $\angle ACB = 5\alpha$, $\angle AED = 4\alpha$. Also $\angle EAC = \alpha$. Let $AB = x$.

$$\begin{aligned} \text{Then } x &= AC \sin 5\alpha = \frac{h \sin (90^\circ + 4\alpha)}{\sin \alpha} \cdot \sin 5\alpha \\ &= h \operatorname{cosec} \alpha \cos 4\alpha \sin 5\alpha. \end{aligned}$$

$$CB = x \cot 5\alpha = h \operatorname{cosec} \alpha \cos 4\alpha \cos 5\alpha.$$

In the particular case

$$\alpha = 7^\circ 19', \quad 4\alpha = 29^\circ 16', \quad 5\alpha = 36^\circ 35'.$$

$$\text{Then } x = \frac{39 \cos 29^\circ 16' \times \sin 36^\circ 35'}{\sin 7^\circ 19'}.$$

$$\log 39 = 1.5911$$

$$\log \cos 29^\circ 16' = \overline{1.9407}$$

$$\begin{array}{r} \log \sin 36^\circ 35' = \overline{1.7753} \\ \hline 1.3071 \end{array}$$

$$\log \sin 7^\circ 19' = \overline{1.1050}$$

$$\log x = 2.2021$$

$\therefore x = 159.2$; thus the height is 159.2 ft.

Again,

$$CB = \frac{x}{\tan 5\alpha} = \frac{x}{\tan 36^\circ 35'}.$$

$$\log x = 2.2021$$

$$\log \tan 36^\circ 35' = \overline{1.8705}$$

$$\log CB = 2.3316$$

Thus the distance is 214.5 ft.

EXAMPLES. XVIII. a. PAGE 206.

1. Area = $\frac{1}{2} \times 300 \times 120 \sin 150^\circ = 300 \times 60 \times \frac{1}{2} = 9000$ sq. feet.

2. $2s = 171 + 204 + 195 = 570$;

$$\therefore \text{area} = \sqrt{285 \times 114 \times 81 \times 90} = \sqrt{10^2 \times 57^2 \times 27^2} = 15390.$$

3. Let $a = 70$, $b = 147$, $c = 119$; then $s = 168$.

$$\therefore \sin B = \frac{2}{70 \times 119} \sqrt{168 \times 98 \times 21 \times 49} = \frac{2 \times 12 \times 7 \times 49}{70 \times 119} = \frac{84}{85}.$$

4. Let $a = 39$, $b = 40$, $c = 25$, and denote the perpendiculars by p_1 , p_2 , p_3 .

Then area = $\sqrt{52 \times 13 \times 12 \times 27} = 12 \times 13 \times 3$.

$$\therefore p_1 = \frac{2\Delta}{a} = 24, \quad p_2 = \frac{2\Delta}{b} = \frac{117}{5}, \quad p_3 = \frac{2\Delta}{c} = \frac{936}{25}.$$

5. Area = $\frac{30^2 \sin 22\frac{1}{2}^\circ \sin 112\frac{1}{2}^\circ}{2 \sin 135^\circ} = \frac{30^2 \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ}{2 \sin 45^\circ} = \frac{30^2}{4} = 225$ sq. ft.

6. The diagonal bisects the parallelogram;

$$\therefore \text{area} = 42 \times 32 \sin 30^\circ = 672$$
 sq. feet.

7. Let a yds. be the length of a side.

Then $\text{area} = a^2 \sin 150^\circ = \frac{a^2}{2}$;

$$\therefore \frac{a^2}{2} = 648; \quad \therefore a = 36 \text{ yards.}$$

\therefore length of a side is 36 yds.

8. $13 + 14 + 15 = 42$;

$$\therefore \Delta = \sqrt{21 \times 8 \times 7 \times 6} = 4 \times 3 \times 7.$$

$$\therefore R = \frac{13 \times 14 \times 15}{4\Delta} = \frac{65}{8} = 8\frac{1}{8}.$$

$$r = \frac{\Delta}{21} = 4.$$

9. $17 + 10 + 21 = 48$;

$$\therefore \Delta = \sqrt{24 \times 7 \times 14 \times 3} = 4 \times 3 \times 7.$$

$$\therefore r_1 = \frac{\Delta}{7} = 12, \quad r_2 = \frac{\Delta}{14} = 6, \quad r_3 = \frac{\Delta}{3} = 28.$$

10. $s - a = \frac{\Delta}{r_1} = 12; \quad s - b = \frac{\Delta}{r_2} = 8; \quad s - c = \frac{\Delta}{r_3} = 4; \quad \therefore s = 24;$

$$\therefore a = 12, \quad b = 16, \quad c = 20.$$

$$11. \quad \sqrt{rr_1r_2r_3} = \sqrt{\frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}} = \Delta.$$

$$12. \quad s(s-a) \tan \frac{A}{2} = s(s-a) \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{s(s-a)(s-b)(s-c)} = \Delta.$$

$$13. \quad rr_1 \cot \frac{A}{2} = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{s(s-a)}{\Delta} = \Delta.$$

$$14. \quad 4Rrs = \frac{abc}{\Delta} \cdot \frac{\Delta}{s} \cdot s = abc.$$

$$15. \quad r_1r_2r_3 = \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{\Delta^3 s}{\Delta^2} = \Delta s = rs^2.$$

$$16. \quad r \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \cdot \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} = r_1.$$

$$17. \quad \text{First side} = r \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) = rs = \Delta.$$

$$18. \quad r_1r_2 + rr_3 = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{s(s-c)} = \frac{\Delta^2 \{s(s-c) + (s-a)(s-b)\}}{\Delta^2} \\ = 2s^2 - (a+b+c)s + ab = ab.$$

$$20. \quad r_1 + r_2 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \\ = 4R \cos \frac{C}{2} \cdot \sin \frac{A+B}{2} = 4R \cos^2 \frac{C}{2} \\ = \frac{2c}{\sin C} \cdot \cos^2 \frac{C}{2} = c \cot \frac{C}{2}.$$

$$21. \quad \text{As in Ex. 20, } r_1 - r = 4R \sin \frac{A}{2} \cdot \cos \frac{B+C}{2} = 4R \sin^2 \frac{A}{2}, \\ r_2 + r_3 = 4R \cos^2 \frac{A}{2}. \\ \therefore (r_1 - r)(r_2 + r_3) = 4R^2 \sin^2 A = a^2.$$

22. By the formulæ of Art. 212, we have

$$\frac{r_1}{r} = \cot \frac{B}{2} \cot \frac{C}{2}; \quad \therefore r_1 \cot \frac{A}{2} = r \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$23. \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a+s-b+s-c}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

24. $r_2 r_3 + r_3 r_1 + r_1 r_2 = \frac{\Delta^2}{(s-b)(s-c)} + \dots + \dots = s(s-a) + \dots + \dots$
 $= 3s^2 - (a+b+c)s = s^2.$

25. As in Ex. 21, $r_2 + r_3 = 4R \cos^2 \frac{A}{2}$, $r_1 - r = 4R \sin^2 \frac{A}{2}$.

$$\therefore r_1 + r_2 + r_3 - r = 4R \left(\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} \right) = 4R.$$

26. $r_1 + r_2 = 4R \cos^2 \frac{C}{2}$, $r_3 - r = 4R \sin^2 \frac{C}{2}$. [Ex. 21.]
 $\therefore r + r_1 + r_2 - r_3 = 4R \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) = 4R \cos C.$

27. $b^2 \sin 2C + c^2 \sin 2B = 2b \sin C \cdot b \cos C + 2c \sin B \cdot c \cos B$
 $= 2b \sin C (b \cos C + c \cos B) = 2ab \sin C = 4\Delta.$

28. $(a+b) \sec \frac{A-B}{2} = 2R (\sin A + \sin B) \sec \frac{A-B}{2}$
 $= 4R \sin \frac{A+B}{2} \cos \frac{A-B}{2} \sec \frac{A-B}{2} = 4R \cos \frac{C}{2}.$

29. $a^2 - b^2 = 4R^2 (\sin^2 A - \sin^2 B) = 4R^2 \sin(A+B) \sin(A-B)$
 $= 4R^2 \sin C \sin(A-B) = 2Rc \sin(A-B).$

30. First side $= 4R^2 \cdot \frac{\sin^2 A - \sin^2 B}{2} \cdot \frac{\sin A \sin B}{\sin(A-B)}$
 $= \frac{4R^2 \sin(A+B) \sin A \sin B}{2}$
 $= \frac{2R \sin A \cdot 2R \sin B \cdot \sin C}{2} = \frac{ab}{2} \sin C = \Delta.$

31. (1) We have $2\Delta = ap_1 = bp_2 = cp_3$;

$$\therefore \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

$$(2) \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a+b-c}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}.$$

32. $(r_1 - r)(r_2 - r)(r_3 - r) = 64R^3 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}$ [Ex. 21]
 $= 4Rr^2.$ [Art. 212.]

$$33. \quad \left(\frac{1}{r} - \frac{1}{r_1}\right) \left(\frac{1}{r} - \frac{1}{r_2}\right) \left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{(r_1 - r)(r_2 - r)(r_3 - r)}{r^3 r_1 r_2 r_3} = \frac{4Rr^2}{r^2 \Delta^2} \quad [\text{Ex. 11}]$$

$$= \frac{4R}{r^2 s^2}.$$

$$34. \quad 4\Delta (\cot A + \cot B + \cot C) = 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$$

$$= a^2 + b^2 + c^2.$$

$$35. \quad \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = \frac{(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)}{\Delta}$$

$$= \frac{s(b-c+c-a+a-b) - \{a(b-c) + b(c-a) + c(a-b)\}}{\Delta} = 0.$$

$$36. \quad a^2 b^2 c^2 (\sin 2A + \sin 2B + \sin 2C) = 4a^2 b^2 c^2 \sin A \sin B \sin C$$

$$= 4 \cdot bc \sin A \cdot ca \sin B \cdot ab \sin C$$

$$= 32\Delta^3.$$

$$37. \quad a \cos A + b \cos B + c \cos C = 2R (\sin A \cos A + \sin B \cos B + \sin C \cos C)$$

$$= R (\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C.$$

$$38. \quad a \cot A + b \cot B + c \cot C = 2R (\cos A + \cos B + \cos C)$$

$$= 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

$$= 2(R+r).$$

$$39. \quad (b+c) \tan \frac{A}{2} = 2R (\sin B + \sin C) \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = 4R \cos \frac{B-C}{2} \cos \frac{B+C}{2}$$

$$= 2R (\cos B + \cos C);$$

$$\therefore (b+c) \tan \frac{A}{2} + \text{two similar terms} = 4R (\cos A + \cos B + \cos C).$$

$$40. \quad r(\sin A + \sin B + \sin C) = 4r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 16R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$= 2R \sin A \sin B \sin C.$$

$$\begin{aligned}
 41. \quad & \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = \frac{1}{2}(3 + \cos A + \cos B + \cos C) \\
 & = \frac{1}{2} \left(4 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
 & = 2 + \frac{r}{2R}.
 \end{aligned}$$

EXAMPLES. XVIII. b. PAGE 210.

1. By Art. 214, the area = $\frac{1}{2} \cdot 10 \cdot 9 \cdot \sin 36^\circ$
 $= 45 \times .588 = 26.46$ sq. ft.

2. By Art. 215, the perimeter = $2 \cdot 15 \cdot \frac{3}{2} \cdot \tan 12^\circ$
 $= 45 \times .213 = 9.585$ yds.

The area = $15 \cdot \frac{9}{4} \cdot \tan 12^\circ = \frac{135 \times .213}{4} = 7.18875$ sq. yds.

3. In fig. of Art. 215 let $AB = 2x$ = side of regular hexagon. Then
 $OD = x \cot 30^\circ = x\sqrt{3}$.
 \therefore area of inscribed circle = $3\pi x^2$.

Again, in fig. of Art. 214, if AB be the side of the hexagon,
 $OA = AD \operatorname{cosec} 30^\circ = 2x$.
 \therefore area of circumscribed circle = $4\pi x^2$.
 \therefore ratio of areas is 3 to 4.

4. With fig. and notation of Art. 217,
area of pentagon = $5AD \cdot OD = 5r^2 \tan 36^\circ$;
 $\therefore 250 = 5r^2 \tan 36^\circ$; $\pi r^2 = \frac{22}{7} \times 50 \cot 36^\circ = 216.23$ sq. ft.

5. Area of circle = $\pi r^2 = 1386$; whence $r = 21$.

By Art. 214, perimeter = $16r \sin 22^\circ 30' = 16 \times 21 \times .382 = 128.352$ in.

6. Area of circle = $\pi r^2 = 616$; whence $r = 14$.

By Art. 215, perimeter of pentagon = $2 \cdot 5 \cdot 14 \tan 36^\circ = 140 \times .727$
 $= 101.78$ ft.

7. Area of circle = $\pi r^2 = 2464$; whence $r = 28$.

Now in Art. 214, if AB is a side of the quindecagon, $OD = 28$, and diameter of required circle = $2AO = 2OD \sec 12^\circ = 2 \times 28 \times 1.022 = 57.232$ ft.

8. Let r be the radius of the circle; then

for the pentagon, $50 = \frac{5}{2} r^2 \sin \frac{2\pi}{5}$,

for the dodecagon, $\text{area} = \frac{12}{2} r^2 \sin \frac{\pi}{6}$; [Art. 214]

$$\therefore \frac{\text{area of dodecagon}}{50} = \frac{12}{5} \cdot \frac{\sin 30^\circ}{\sin 72^\circ};$$

$$\therefore \text{area of dodecagon} = 60 \operatorname{cosec} 72^\circ = 60 \times 1.0515 = 63.09 \text{ sq. ft.}$$

9. Let the perimeters of pentagon and decagon be denoted by $10a$ and $10b$ respectively. Then as in Example 2, page 209,

$$\text{area of pentagon} = 5a^2 \cot \frac{\pi}{5},$$

$$\text{area of decagon} = \frac{10b^2}{4} \cot \frac{\pi}{10};$$

$$\therefore 2a^2 \cot 36^\circ = b^2 \cot 18^\circ.$$

Now $\cot^2 18^\circ = \frac{1 + \cos 36^\circ}{1 - \cos 36^\circ} = \frac{1 + \frac{\sqrt{5}+1}{4}}{1 - \frac{\sqrt{5}+1}{4}} = \frac{5+\sqrt{5}}{3-\sqrt{5}}$;

and $\cot^2 36^\circ = \frac{1 + \frac{\sqrt{5}-1}{4}}{1 - \frac{\sqrt{5}-1}{4}} = \frac{3+\sqrt{5}}{5-\sqrt{5}}$;

$$\therefore \frac{\cot^2 18^\circ}{\cot^2 36^\circ} = \frac{20}{4} = 5.$$

$$\therefore \frac{a^2}{b^2} = \frac{\sqrt{5}}{2}, \text{ or } \frac{a}{b} = \frac{\sqrt[4]{5}}{\sqrt{2}}.$$

10. Let $2na$ be the common perimeter, so that $2a, a$ are sides respectively of the two polygons.

$$\text{Area of polygon of } n \text{ sides} = \frac{n}{4} \cdot 4a^2 \cot \frac{\pi}{n}.$$

$$\text{Area of polygon of } 2n \text{ sides} = \frac{2n}{4} \cdot a^2 \cot \frac{\pi}{2n};$$

$$\begin{aligned}\therefore \text{ratio of areas} &= \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left(\sin \frac{\pi}{n} \cos \frac{\pi}{2n} \right) \\ &= \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left(2 \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n} \right) \\ &= \left(2 \cos \frac{\pi}{n} \right) : \left(1 + \cos \frac{\pi}{n} \right).\end{aligned}$$

11. In the fig. of Art. 214, if $AD=a$, $OA=R$, we have $R=a \operatorname{cosec} \frac{\pi}{n}$.

In the fig. of Art. 215, if $OD=r$, we have $r=a \cot \frac{\pi}{n}$.

$$\begin{aligned}\therefore R+r &= a \left(\frac{1}{\sin \frac{\pi}{n}} + \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right) \\ &= \frac{a \left(1 + \cos \frac{\pi}{n} \right)}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = \frac{2a \cos^2 \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = a \cot \frac{\pi}{2n}.\end{aligned}$$

12. Let p , h , d represent a side of the pentagon, hexagon, and decagon respectively inscribed in a circle of radius r ; then

$$p=2r \sin 36^\circ, \quad h=2r \sin 30^\circ, \quad d=2r \sin 18^\circ.$$

$$\begin{aligned}\therefore h^2+d^2 &= 4r^2 \left\{ \frac{1}{4} + \left(\frac{\sqrt{5}-1}{4} \right)^2 \right\} = 4r^2 \left(\frac{4+6-2\sqrt{5}}{16} \right) \\ &= 4r^2 \left(\frac{10-2\sqrt{5}}{16} \right) = 4r^2 \sin^2 36^\circ = p^2. \quad [\text{See Ex. Art. 126.}]\end{aligned}$$

$$13. \quad A_1 = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}, \quad B_1 = nr^2 \tan \frac{\pi}{n}, \quad [\text{Arts. 214, 215}]$$

$$A_2 = \frac{1}{2} 2n \cdot r^2 \sin \frac{\pi}{n}, \quad B_2 = 2nr^2 \tan \frac{\pi}{2n}.$$

$$\therefore A_1 B_1 = \frac{1}{2} n^2 r^4 \cdot 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \tan \frac{\pi}{n} = n^2 r^4 \sin^2 \frac{\pi}{n} = A_2^2.$$

Thus A_2 is the geom. mean between A_1 and B_1 .

$$\begin{aligned}\text{Again } \frac{1}{A_2} + \frac{1}{B_1} &= \frac{1}{nr^2 \sin \frac{\pi}{n}} + \frac{1}{nr^2 \tan \frac{\pi}{n}} = \frac{1 + \cos \frac{\pi}{n}}{nr^2 \sin \frac{\pi}{n}} \\ &= \frac{2 \cos^2 \frac{\pi}{2n}}{2nr^2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} = \frac{1}{nr^2 \tan \frac{\pi}{2n}} = \frac{2}{B_2}.\end{aligned}$$

Thus B_2 is the harm. mean between A_2 and B_1 .

EXAMPLES. XVIII. c. PAGE 218.

1. Required distance

$$= r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$2. I_1 A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2},$$

$$I_1 B = r_1 \operatorname{cosec} \left(90^\circ - \frac{B}{2} \right) = r_1 \sec \frac{B}{2} = 4R \sin \frac{A}{2} \cos \frac{C}{2},$$

$$I_1 C = r_1 \operatorname{cosec} \left(90^\circ - \frac{C}{2} \right) = r_1 \sec \frac{C}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2}.$$

3. (1) From the fig. of Art. 219, we have

$$\begin{aligned} \text{area} &= \frac{1}{2} I_1 I_2 \cdot I_3 C = \frac{1}{2} 4R \cos \frac{C}{2} \cdot r_3 \operatorname{cosec} \frac{C}{2} \\ &= 2R r_3 \cot \frac{C}{2} = 2Rs. \end{aligned}$$

$$(2) \text{ Area} = 2Rs = 2R \cdot \frac{\Delta}{r} = \frac{1}{2} \Delta \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}. \quad [\text{Art. 212.}]$$

4. We have $II_1 = IC \operatorname{cosec} II_1 C = IC \operatorname{cosec} \frac{B}{2}$.

$$\begin{aligned} \therefore rII_1 \cdot II_2 \cdot II_3 &= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot IA \cdot IB \cdot IC \cdot \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \\ &= 4R \cdot IA \cdot IB \cdot IC. \end{aligned}$$

5. Perimeter of pedal triangle $= R (\sin 2A + \sin 2B + \sin 2C)$
 $= 4R \sin A \sin B \sin C$.

In-radius of pedal triangle

$$\begin{aligned} &= 4 \cdot \frac{R}{2} \sin (90^\circ - A) \sin (90^\circ - B) \sin (90^\circ - C) \quad [\text{Arts. 225, 212}] \\ &= 2R \cos A \cos B \cos C. \end{aligned}$$

$$\begin{aligned} 6. (1) \quad \frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} &= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \\ &= \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{2abc} \\ &= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}. \end{aligned}$$

$$(2) \quad \frac{b^2 - c^2}{a^2} g + \dots + \dots = \frac{b^2 - c^2}{a} \cos A + \dots + \dots \\ = \frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2abc} + \dots + \dots \\ = 0.$$

7. Let ρ_1, ρ_2, ρ_3 be the radii, then

$$\rho_1 = 4 \frac{R}{2} \cdot \sin \frac{G}{2} \cos \frac{H}{2} \cos \frac{K}{2} = 2R \cos A \sin B \sin C. \quad [\text{Arts. 225, 212.}]$$

8. (1) $a \cos A, b \cos B, c \cos C, 180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$ are the sides and angles of the pedal triangle of the triangle ABC . Hence in any formula connecting a, b, c, A, B, C , we may replace the sides by $a \cos A, b \cos B, c \cos C$ respectively, and the angles by $180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$ respectively.

(2) $a \operatorname{cosec} \frac{A}{2}, b \operatorname{cosec} \frac{B}{2}, c \operatorname{cosec} \frac{C}{2}, 90^\circ - \frac{A}{2}, 90^\circ - \frac{B}{2}, 90^\circ - \frac{C}{2}$ are the sides and angles of the ex-central triangle of the triangle ABC . Hence the proposition follows.

9. We have $SI^2 = R^2 - 2Rr$ [Art. 228].

$\therefore R^2 - 2Rr$ must be positive; that is $R - 2r$ must be positive.

Hence R can never exceed $2r$.

10. If

$$R = 2r;$$

we have

$$SI^2 = R^2 - 2Rr = 0;$$

\therefore the centres of the circumcircle and in-circle coincide, and hence the triangle is equilateral.

$$11. \quad SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = R^2 - 2Rr + R^2 + 2Rr_1 + R^2 + 2Rr_2 + R^2 + 2Rr_3 \\ = 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\ = 12R^2 \quad [\text{XVIII. a. Ex. 25.}]$$

$$12. \quad (1) \quad a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = r^2 a \operatorname{cosec}^2 \frac{A}{2} + \dots + \dots$$

$$= abcr^2 \left\{ \frac{1}{(s-b)(s-c)} + \dots + \dots \right\} \\ = \frac{abcr^2 s}{(s-a)(s-b)(s-c)} = \frac{abcr^2 s^2}{\Delta^2} = abc.$$

$$(2) \quad a \cdot AI_1^2 - b \cdot BI_1^2 - c \cdot CI_1^2 = r_1^2 \left(a \operatorname{cosec}^2 \frac{A}{2} - b \sec^2 \frac{B}{2} - c \sec^2 \frac{C}{2} \right)$$

$$= abcr_1^2 \left\{ \frac{1}{(s-b)(s-c)} - \frac{1}{s(s-b)} - \frac{1}{s(s-c)} \right\}$$

$$= \frac{abcr_1^2 \{s - (s-c) - (s-b)\}}{s(s-b)(s-c)} = \frac{abcr_1^2 (s-a)^2}{\Delta^2} = abc.$$

13. (1) We have $\frac{OG}{AG} = \frac{\Delta OBC}{\Delta ABC}$.

$$\therefore \frac{OG}{AG} + \frac{OH}{BH} + \frac{OK}{CK} = \frac{\text{sum of areas of } \Delta^* OBC, OAC, OAB}{\Delta ABC} = 1.$$

(2) Since A, H, O, K are concyclic, $HK = AO \sin A$;

$$\text{thus } AO = \frac{a \cos A}{\sin A} = a \cot A.$$

$\therefore OG + a \cot A = OG + AO = AG$, and the result required is reduced to that already proved in (1).

14. Circum-radius of $\Delta AHK = \frac{KH}{2 \sin A} = \frac{R \sin 2A}{2 \sin A} = R \cos A$;

\therefore sum of circum-radii of $\Delta^* AHK, BKG, CGH = R(\cos A + \cos B + \cos C)$

$$= R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= R + r.$$

15. We have $A_1 = \frac{\pi}{2} - \frac{A}{2}$,

$$A_2 = \frac{\pi}{2} - \frac{A_1}{2} = \frac{\pi}{2} - \frac{1}{2} \left(\frac{\pi}{2} - \frac{A}{2} \right),$$

$$A_3 = \frac{\pi}{2} - \frac{A_2}{2} = \frac{\pi}{2} - \frac{1}{2} \left\{ \frac{\pi}{2} - \frac{1}{2} \left(\frac{\pi}{2} - \frac{A}{2} \right) \right\};$$

.....

$$\therefore A_n = \frac{\pi}{2} \left(1 - \frac{1}{2} + \left(\frac{1}{2} \right)^2 - \dots \text{to } n \text{ terms} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{2} \frac{1 - \left(-\frac{1}{2} \right)^n}{1 + \frac{1}{2}} + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} \left(1 - (-1)^n \frac{1}{2^n} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(A - \frac{\pi}{3} \right);$$

similarly $B_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(B - \frac{\pi}{3} \right),$

$$C_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(C - \frac{\pi}{3} \right).$$

When n is indefinitely increased, then $A_n = B_n = C_n = \frac{\pi}{3}$.

16. (1) $OS^2 = R^2 - 8R^2 \cos A \cos B \cos C$ [Art. 230]
 $= R^2 + 2R^2 (1 + \cos 2A + \cos 2B + \cos 2C)$
 $= 9R^2 - 2R^2 (1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C)$
 $= 9R^2 - 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C)$
 $= 9R^2 - a^2 - b^2 - c^2.$

(2) We have $AO = 2R \cos A; IA = r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2};$

$$\angle IAO = (90^\circ - B) - \frac{A}{2} = \frac{C - B}{2};$$

$$\begin{aligned}OI^2 &= 4R^2 \cos^2 A + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 16R^2 \cos A \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{C - B}{2} \\&= 4R^2 \left(\cos^2 A + 4 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C - 4 \cos A \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \right) \\&= 4R^2 \left(\cos^2 A + 8 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\&= 2r^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\&= 2r^2 - 4R^2 \cos A \cos B \cos C.\end{aligned}$$

(3) We have

$$AO = 2R \cos A; IA_1 = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}; \angle I_1 AO = \frac{C - B}{2};$$

$$\begin{aligned}OI_1^2 &= 4R^2 \cos^2 A + 16R^2 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 16R^2 \cos A \cos \frac{B}{2} \cos \frac{C}{2} \cos \frac{C - B}{2} \\&= 4R^2 \left(\cos^2 A + 4 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 4 \cos A \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\&= 4R^2 \left(\cos^2 A + 8 \sin^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\&= 2r_1^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\&= 2r_1^2 - 4R^2 \cos A \cos B \cos C.\end{aligned}$$

17. Let R' , ρ , ρ_1 , ρ_2 , ρ_3 represent the circum-radius, the in-radius, and the ex-radii of the pedal triangle; then, since A , B , C are the ex-centres of the pedal triangle, we have, as in Art. 229,

$$j^2 = R'^2 + 2R'\rho_1; \text{ also } R' = \frac{R}{2}.$$

$$\therefore j^2 + g^2 + h^2 = 3R'^2 + 2R'(\rho_1 + \rho_2 + \rho_3)$$

[XVIII. a. Ex. 25]

$$= 3R'^2 + 2R'(4R' + \rho)$$

$$= 11R'^2 + 2R'\rho$$

$$= 11R'^2 + 8R'^2 \sin \frac{G}{2} \sin \frac{H}{2} \sin \frac{F}{2}$$

$$= 11R'^2 + 8R'^2 \cos A \cos B \cos C;$$

[Art. 224]

$$\therefore 4(j^2 + g^2 + h^2) = 11R^2 + 8R^2 \cos A \cos B \cos C.$$

[Art. 225.]

EXAMPLES. XVIII. d. PAGE 223.

1. Let r be the radius, and a , b , c , d the sides of the quadrilateral.

Then we have $2S = ra + rb + rc + rd$;

$$\therefore r = \frac{S}{\sigma}.$$

2. Let $ABCD$ be the quadrilateral having sides

$$AB = BC = 3; \quad CD = DA = 4.$$

Then $\angle BAC = \angle BCA; \quad \angle DAC = \angle DCA;$

$$\angle BAD + \angle BCD = 180^\circ; \quad \therefore \angle BAD = \angle BCD = 90^\circ,$$

and the $\triangle^* BAD, BCD$ are identically equal.

Thus it easily follows that the bisectors of the $\angle^* BAD, BCD$ meet on BD which bisects the angles ABC, ADC .

\therefore a circle can be inscribed in the quadrilateral.

If r be its radius we have

$$r(3+3+4+4)=2 \text{ (area of quadrilateral)}=2 \cdot 4 \cdot 3.$$

$$\therefore r = \frac{12}{7} = 1\frac{5}{7}.$$

Also radius of circumscribed circle = $\frac{1}{2}BD = \frac{1}{2}\sqrt{3^2 + 4^2} = \frac{5}{2} = 2\frac{1}{2}$.

3. Let $ABCD$ be the quadrilateral, and let

$$AB=1, BC=2, CD=4, DA=3.$$

Then

$$AC^2=5-4 \cos ABC=5+4 \cos ADC;$$

also

$$AC^2=25-24 \cos ADC;$$

$$\therefore 5+4 \cos ADC=25-24 \cos ADC.$$

$$\therefore \cos ADC=\frac{20}{28}=\frac{5}{7};$$

and area of quadrilateral = $\sqrt{4 \times 3 \times 1 \times 2} = \sqrt{24}$;

$$\therefore \text{radius of inscribed circle} = \frac{\sqrt{24}}{5} = .98 \text{ nearly.}$$

4. Let $ABCD$ be the quadrilateral, and let

$$AB=60, BC=25, CD=52, DA=39.$$

$$\text{Then } AC^2=60^2+25^2-2 \cdot 60 \cdot 25 \cos ABC=5^2 \times 13^2-2 \times 60 \times 25 \cos ABC;$$

$$\text{also } AC^2=52^2+39^2-2 \times 52 \times 39 \cos ADC=5^2 \times 13^2+2 \times 52 \times 39 \cos ABC;$$

$$\therefore \cos ABC=0, \text{ that is } \angle ABC=90^\circ;$$

and hence the $\angle ADC=90^\circ$;

$$\therefore AC^2=60^2+25^2=5^2 \times 13^2; \therefore AC=65;$$

$$\therefore BD=\frac{60 \times 52+39 \times 25}{65}=48+15=63.$$

Also the area = $\sqrt{28 \times 63 \times 36 \times 49}=6^2 \times 7^2=1764$.

5. Let $ABCD$ be the quadrilateral, and let

$$AB=4, BC=5, CD=8, DA=9.$$

Then since $AB+BC=9$, AC must be less than 9;

\therefore diagonal $BD=9$;

$$\text{and } \cos A=\frac{2}{9}; \therefore \sin A=\frac{\sqrt{77}}{9};$$

$$\cos C=\frac{1}{10}; \therefore \sin C=\frac{3\sqrt{11}}{10};$$

$$\therefore \text{area}=\frac{1}{2}(ad \sin A+bc \sin C)=18\frac{\sqrt{77}}{9}+20 \cdot \frac{3\sqrt{11}}{10}=2\sqrt{77}+6\sqrt{11}.$$

6. Since the quadrilateral is such that one circle can be inscribed in it and another circle circumscribed about it, therefore

$$\cos A = \frac{ad - bc}{ad + bc}, \text{ and } \sin A = \frac{2\sqrt{abcd}}{ad + bc}; \quad [\text{Art. 234, Ex.}]$$

$$\therefore \text{area} = \frac{1}{2}(ad \sin A + bc \sin C) = \sqrt{abcd}.$$

If r be radius of inscribed circle,

$$\frac{r}{2}(a+b+c+d) = \text{area} = \sqrt{abcd}.$$

$$\therefore r = \frac{2\sqrt{abcd}}{a+b+c+d}.$$

7. We have $S^2 = (\sigma - a)(\sigma - b)(\sigma - c)(\sigma - d) - abcd \cos^2 a$; [Art. 232]

$\therefore S$ is greatest when $\cos a = 0$, since σ, a, b, c, d are constant.

But a is half the sum of two opposite angles;

\therefore area is a maximum when the sum of two opposite angles is 180° ; that is, when the quadrilateral can be inscribed in a circle.

8. $23 + 29 + 37 + 41 = 130$;

$$\therefore \text{maximum area} = \sqrt{(65 - 23)(65 - 29)(65 - 37)(65 - 41)} \text{ sq. inches} \\ = 6 \times 6 \times 4 \times 7 \text{ sq. inches} = 7 \text{ sq. feet.}$$

9. We have $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$; [Art. 233]

$$\therefore \tan^2 \frac{B}{2} = \frac{1 - \cos B}{1 + \cos B} = \frac{(c+d)^2 - (a-b)^2}{(a+b)^2 - (c-d)^2} \\ = \frac{(a+c+d-b)(c+d+b-a)}{(a+b+c-d)(a+b+d-c)} = \frac{(\sigma - a)(\sigma - b)}{(\sigma - c)(\sigma - d)}.$$

10. See figure on page 220. Let $\angle DPA = \beta$;

then $a^2 = AP^2 + PB^2 + 2AP \cdot PB \cos \beta$;

$$c^2 = DP^2 + PC^2 + 2DP \cdot PC \cos \beta;$$

$$b^2 = BP^2 + PC^2 - 2BP \cdot PC \cos \beta;$$

$$d^2 = DP^2 + PA^2 - 2DP \cdot PA \cos \beta;$$

$$\therefore (a^2 + c^2) - (b^2 + d^2) = 2 \cos \beta \{ AP(DP + PB) + PC(DP + PB) \} \\ = 2AC \cdot BD \cos \beta = 2fg \cos \beta.$$

11. Area = $\frac{1}{2} AC \cdot BD \sin \beta$ [Art. 231]

$$= \frac{1}{4} \{(a^2 + c^2) \sim (b^2 + d^2)\} \tan \beta. \quad [\text{Ex. 10.}]$$

12. We have $a+c=b+d; \therefore (a-d)^2=(b-c)^2.$

Also $a^2+d^2-2ad \cos A=b^2+c^2-2bc \cos C;$

\therefore by subtraction, $ad(1-\cos A)=bc(1-\cos C);$

that is, $ad-bc=ad \cos A - bc \cos C;$

$$\therefore a^2d^2(1-\cos^2 A)+b^2c^2(1-\cos^2 C)=2abcd-2abcd \cos A \cos C;$$

or $a^2d^2 \sin^2 A+b^2c^2 \sin^2 C=2abcd-2abcd \cos A \cos C.$

$$\therefore (2S)^2-2abcd \sin A \sin C=2abcd-2abcd \cos A \cos C.$$

$$\therefore 4S^2=2abcd(1-\cos \overline{A+C})=4abcd \sin^2 \frac{A+C}{2}.$$

$$\therefore S=\sqrt{abcd} \sin \frac{A+C}{2}.$$

13. If β be the angle between the diagonals, we have

$$S=\frac{1}{2}fg \sin \beta.$$

$$\therefore f^2g^2-4S^2=f^2g^2 \cos^2 \beta$$

$$= \frac{1}{4} \{(a^2+c^2)-(b^2+d^2)\}^2 \quad [\text{Ex. 10.}]$$

$$= \frac{1}{4} (2bd-2ac)^2, \text{ since } a+c=b+d;$$

that is,

$$4S^2=f^2g^2-(ac-bd)^2.$$

14. (1) By Euc. vi. D, we have

$$(ac+bd) \sin \beta=fg \sin \beta=2S=(ad+bc) \sin A.$$

(2) $\cos \beta=\frac{(a^2+c^2) \sim (b^2+d^2)}{2fg}$ [Ex. 10.]

$$=\frac{(a^2+c^2) \sim (b^2+d^2)}{2(ac+bd)}.$$

$$(3) \quad \tan^2 \frac{\beta}{2} = \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{(b+d)^2 - (c-a)^2}{(c+a)^2 - (b-d)^2}, \text{ or } \frac{(c+a)^2 - (b-d)^2}{(b+d)^2 - (c-a)^2}$$

$$= \frac{(b+d+c-a)(b+d-c+a)}{(c+a-b+d)(c+a+b-d)}, \text{ or } \frac{(c+a-b+d)(c+a+b-d)}{(b+d+c-a)(b+d-c+a)}$$

$$= \frac{(\sigma-a)(\sigma-c)}{(\sigma-b)(\sigma-d)}, \text{ or } \frac{(\sigma-b)(\sigma-d)}{(\sigma-a)(\sigma-c)}.$$

15. $S = \frac{1}{2}fg \sin \beta$

$$= \frac{1}{4} \sqrt{4f^2g^2 - 4f^2g^2 \cos^2 \beta}$$

$$= \frac{1}{4} \sqrt{4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2}. \quad [\text{Ex. 10.}]$$

16. See figure on page 220.

We have $\frac{AP}{PC} = \frac{\Delta APB}{\Delta CPB} = \frac{\Delta APD}{\Delta CPD} = \frac{\Delta DAB}{\Delta DCB} = \frac{ad}{bc};$

$$\therefore AP = \frac{ad}{ad+bc} \cdot AC;$$

$$PC = \frac{bc}{ad+bc} \cdot AC;$$

$$\therefore AP \cdot PC = \frac{abcd}{(ad+bc)^2} \cdot AC^2 = \frac{abcd}{(ad+bc)^2} \cdot \frac{(ad+bc)(ac+bd)}{ab+cd}$$

$$= \frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}.$$

EXAMPLES. XVIII. e. PAGE 225.

1. $242 + 1212 + 1450 = 2904;$

$$\therefore \text{area} = \sqrt{1452 \times 1210 \times 240 \times 2} \text{ sq. yds.}$$

$$= 8 \times 3 \times 121 \times 10 \text{ sq. yds.}$$

$$= \frac{8 \times 3 \times 121 \times 10}{4840} = 6 \text{ acres.}$$

2. $\text{Area} = \frac{200^\circ \sin 22\frac{1}{2}^\circ \sin 67\frac{1}{2}^\circ}{2 \sin (22\frac{1}{2}^\circ + 67\frac{1}{2}^\circ)} = \frac{200^\circ \cos 45^\circ}{4}$

$$= \frac{10000\sqrt{2}}{2} = \frac{14142}{2} = 7071 \text{ sq. yds.}$$

3. We have $\frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{2\Delta}{s-c}$;

$$\therefore b=c, \text{ and } s-b=2(s-a);$$

$$\therefore \frac{c+a-b}{2} = b+c-a;$$

that is,

$$\frac{a}{2} = 2b-a;$$

that is,

$$3a=4b.$$

4. If a, b, c are in A.P., we have $a+c=2b$;

$$\therefore \frac{1}{r_1} + \frac{1}{r_3} = \frac{s-a+s-c}{\Delta} = \frac{2(s-b)}{\Delta} = \frac{2}{r_2};$$

$\therefore r_1, r_2, r_3$ are in H.P.

5. We have $s = \frac{z}{x} + \frac{x}{y} + \frac{y}{z}$.

$$\therefore \text{area} = \sqrt{\left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right) \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = \sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}.$$

6. We have $\frac{\sin(A-B)}{\sin(B-C)} = \frac{\sin A}{\sin C} = \frac{\sin(B+C)}{\sin(A+B)}$;

$$\therefore \sin(A-B)\sin(A+B) = \sin(B-C)\sin(B+C);$$

or

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$$

$$\therefore a^2 - b^2 = b^2 - c^2;$$

that is, a^2, b^2, c^2 are in A.P.

7. First side $= \frac{a \sin A + b \sin B + c \sin C}{\sin A + \sin B + \sin C} = \frac{a^2 + b^2 + c^2}{a+b+c} = \frac{a^2 + b^2 + c^2}{2s}$.

8. First side $= \frac{a}{\sin A} \cdot (a+b+c) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
 $= \frac{abc}{2\Delta} \cdot 2s \frac{(s-a)(s-b)(s-c)}{abc} = \frac{\Delta^2}{\Delta} = \Delta$.

9. By Ex. 20, XVIII. a. we have

$$\begin{aligned} \text{first side} &= abc \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \\ &= abc \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)}} \\ &= \frac{abcs^2}{\Delta} = 4Rs^2 = 4R(r_2r_3 + r_3r_1 + r_1r_2). \end{aligned} \quad [\text{XVIII. a. Ex. 24.}]$$

10. $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{r_1}{s} + \frac{r_2}{s} + \frac{r_3}{s} = \frac{r_1 + r_2 + r_3}{s}$

$$= \frac{r_1 + r_2 + r_3}{(r_2 r_3 + r_3 r_1 + r_1 r_2)^{\frac{1}{2}}}. \quad [\text{XVIII. a. Ex. 24.}]$$

11. First side $= bc \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + ca \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + ab \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$$= \frac{s}{\Delta} \{bc(s-a) + ca(s-b) + ab(s-c)\}$$

$$= \frac{s}{\Delta} \{(bc + ca + ab)s - 3abc\}$$

$$= \frac{abcs^2}{\Delta} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right)$$

$$= 4Rs^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right).$$

12. First side $= \left(\frac{s+s-a+s-b+s-c}{\Delta} \right)^2 = \left(\frac{2s}{\Delta} \right)^2$

$$= \frac{4s}{\Delta} \cdot \left(\frac{s-a+s-b+s-c}{\Delta} \right) = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right).$$

13. We have $\frac{\Delta}{s} = 6$, $2s = 70$; $\therefore \Delta = 6 \times 35$.

Let a, b be the sides containing the right angle;

then $\frac{1}{2}ab = \Delta = 6 \times 35$; $\therefore ab = 12 \times 35$;

also $a^2 + b^2 = c^2 = \{70 - (a+b)\}^2$;

$$\therefore 0 = 70^2 - 140(a+b) + 2ab = 70^2 - 140(a+b) + 12 \times 70;$$

$$\therefore a+b = 41 = 20+21,$$

and $ab = 12 \times 35 = 20 \times 21$;

\therefore the two sides containing the right angle are 20, 21, and the hypotenuse
 $= 70 - 41 = 29$.

14. It is easily seen from a figure that $\frac{a}{2f} = \tan A$;

$$\therefore \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = 2(\tan A + \tan B + \tan C)$$

$$= 2 \tan A \tan B \tan C = \frac{abc}{4fgh}.$$

15. Let r, r' be the radii of the inscribed circles; then since the perimeters of the triangle and hexagon are equal we have

$$6r \tan \frac{\pi}{3} = 12r' \tan \frac{\pi}{6};$$

whence

$$3r = 2r';$$

$$\therefore \frac{\pi r^2}{\pi r'^2} = \frac{4}{9};$$

∴ areas of inscribed circles are as 4 to 9.

16. Perimeter $= R (\sin 2A + \sin 2B + \sin 2C) = 4R \sin A \sin B \sin C$

$$= 4R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{2R^2}.$$

$$\begin{aligned} 17. \text{ Area} &= \frac{1}{2} 4R \cos \frac{B}{2} 4R \cos \frac{C}{2} \sin \left(90^\circ - \frac{A}{2} \right) \\ &= 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\ &= 2R^2 (\sin A + \sin B + \sin C) \\ &= R(a+b+c) = \frac{abc(a+b+c)}{4\Delta}. \end{aligned}$$

18. Let O_1, O_2 be the two circumcentres; then O_1O_2 is at right angles to AC at its middle point. Draw O_1N_1, O_2N_2 perpendicular to AB_1 ;

then

$$O_1O_2 = N_1N_2 \operatorname{cosec} A = \frac{c_1 \sim c_2}{2 \sin A}.$$

19. Let f, g be the diagonals; then by Euc. vi. D.,

$$(ac+bd) \sin \beta = fg \sin \beta = 2S.$$

[Art. 231.]

$$\therefore \sin \beta = \frac{2S}{ac+bd}.$$

20. We have $r \cdot II_1 = r \cdot IC \operatorname{cosec} II_1 C = IC r \operatorname{cosec} \frac{B}{2} = IC \cdot IB$.

$$\therefore r^3 II_1 \cdot II_2 \cdot II_3 = IA^2 \cdot IB^2 \cdot IC^2.$$

21. Sum of squares of sides of ex-central \triangle

$$= 16R^2 \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right)$$

$$= 16R^2 \left(2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \quad [\text{XII. d. Ex. 13}]$$

$$= 32R^2 + 32R^2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 32R^2 + 8Rr = 8R(4R+r).$$

22. Since a circle can be inscribed in the quadrilateral,

$$\therefore a+c=b+d;$$

and since a circle can be circumscribed about the quadrilateral,

$$\begin{aligned}\therefore \cos \beta &= \frac{(a^2 + c^2) \sim (b^2 + d^2)}{2(ac + bd)} && [\text{XVIII. d. Ex. 14}] \\ &= \frac{(a+c)^2 \sim (b+d)^2 + 2(ac \sim bd)}{2(ac + bd)} \\ &= \frac{ac \sim bd}{ac + bd}.\end{aligned}$$

23. (1) We have $2l^2 + 2 \cdot \frac{a^2}{4} = b^2 + c^2$;

$$2m^2 + 2 \cdot \frac{b^2}{4} = c^2 + a^2;$$

$$2n^2 + 2 \cdot \frac{c^2}{4} = a^2 + b^2;$$

\therefore by addition, $4(l^2 + m^2 + n^2) = 3(a^2 + b^2 + c^2)$.

$$\begin{aligned}(2) \quad \text{First side} &= \frac{1}{4} \{ (b^2 - c^2)(2b^2 + 2c^2 - a^2) + \text{two similar terms} \} \\ &= \frac{1}{4} [\{ 2(b^4 - c^4) - a^2(b^2 - c^2) \} + \dots + \dots] \\ &= 0.\end{aligned}$$

$$(3) \quad 4l^2 = 2b^2 + 2c^2 - a^2;$$

$$\therefore 16(l^4 + m^4 + n^4) = (2b^2 + 2c^2 - a^2)^2 + \dots + \dots \\ = 9(a^4 + b^4 + c^4), \text{ on reduction.}$$

24. We have $r_1 + r_2 + r_3 = 4R + r$

[XVIII. a. Ex. 25],

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$$

[XVIII. a. Ex. 24],

$$r_1 r_2 r_3 = rs^2;$$

[XVIII. a. Ex. 15]

\therefore the equation whose roots are r_1, r_2, r_3 is

$$x^3 - (4R + r)x^2 + s^2x - rs^2 = 0.$$

25. Let PQ be the tangent to inscribed circle parallel to BC , and draw AHD at right angles to PQ and BC ;

$$\text{then } \frac{\Delta_1}{\Delta} = \frac{AP \cdot AQ}{AB \cdot AC} = \frac{AH}{AD} \cdot \frac{AH}{AD} = \left(\frac{AD - 2r}{AD} \right)^2$$

$$= \left(1 - \frac{2\Delta}{s} \cdot \frac{a}{2\Delta} \right)^2 = \frac{(s-a)^2}{s^2}.$$

$$\therefore \frac{\Delta_1}{(s-a)^2} = \frac{\Delta}{s^2}$$

$$= \frac{\Delta_2}{(s-b)^2} = \frac{\Delta_3}{(s-c)^2}, \text{ by symmetry.}$$

Otherwise. Let the sides and perimeter of triangle APQ be denoted by $a_1, b_1, c_1, 2s_1$; then, by Art. 213, $s_1 = s - a$.

$$\text{But } \frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{s}{s_1}; \therefore \frac{\Delta_1}{s_1^2} = \frac{\Delta}{s^2}. \quad [\text{Euc. vi. 19}].$$

That is,

$$\frac{\Delta_1}{(s-a)^2} = \frac{\Delta}{s^2}.$$

26. MN is perpendicular to the bisector of the angle A ;

$\therefore MN$ is parallel to I_2I_3 ;

thus the sides of the $\triangle LMN$ are parallel to the sides of the excentral \triangle and the triangles are similar.

$$\text{Also } MN = 2(s-a) \sin \frac{A}{2};$$

$$\begin{aligned} \therefore \frac{\Delta LMN}{\Delta I_1I_2I_3} &= \frac{MN \cdot NL}{I_2I_3 \cdot I_3I_1} = \frac{2(s-a) \sin \frac{A}{2} \cdot 2(s-b) \sin \frac{B}{2}}{4R \cos \frac{A}{2} \cdot 4R \cos \frac{B}{2}} \\ &= \frac{(s-a)(s-b)(s-c)}{4R^2s} = \frac{\Delta^2}{4R^2s^2} = \frac{r^2}{4R^2}. \end{aligned}$$

27. We have $\angle PBC = \angle PCB = A$; $\therefore \angle QPR = 180^\circ - 2A$.

Similarly $\angle PQR = 180^\circ - 2B$, $\angle QRP = 180^\circ - 2C$.

$$\text{Again } AQ = \frac{AC \sin B}{\sin AQC} = \frac{b \sin B}{\sin 2B} = \frac{b}{2 \cos B};$$

similarly

$$AR = \frac{c}{2 \cos C}.$$

$$\therefore QR = \frac{b}{2 \cos B} + \frac{c}{2 \cos C} = \frac{b \cos C + c \cos B}{2 \cos B \cos C} = \frac{a}{2 \cos B \cos C}.$$

28. (1) We have $pc \sin \frac{A}{2} + pb \sin \frac{A}{2} = 2\Delta = bc \sin A$;

$$\therefore p(b+c) = 2bc \cos \frac{A}{2};$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} = \frac{b+c}{2bc} = \frac{1}{2} \left(\frac{1}{b} + \frac{1}{c} \right);$$

$$\text{similarly } \frac{1}{q} \cos \frac{B}{2} = \frac{1}{2} \left(\frac{1}{c} + \frac{1}{a} \right), \quad \frac{1}{r} \cos \frac{C}{2} = \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right).$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} + \frac{1}{q} \cos \frac{B}{2} + \frac{1}{r} \cos \frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$\begin{aligned}
 (2) \quad \text{We have } pqr &= \frac{8a^2b^2c^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{(b+c)(c+a)(a+b)} \\
 &= \frac{2a^2b^2c^2 (\sin A + \sin B + \sin C)}{(b+c)(c+a)(a+b)} \\
 &= \frac{4\Delta abc (a+b+c)}{(b+c)(c+a)(a+b)}; \\
 \therefore \frac{pqr}{4\Delta} &= \frac{abc (a+b+c)}{(b+c)(c+a)(a+b)}.
 \end{aligned}$$

29. Let O be the orthocentre; then G, H, K are the middle points of OL, OM, ON respectively; therefore the sides of the $\triangle LMN$ are double the sides of the pedal triangle and parallel to them.

Therefore also the angles are equal to the angles of the pedal triangle.

$$\begin{aligned}
 (1) \quad \text{Area of } \triangle LMN &= \frac{1}{2} \cdot 2a \cos A \cdot 2b \cos B \cdot \sin (180^\circ - 2C) \\
 &= 4ab \sin C \cdot \cos A \cos B \cos C = 8\Delta \cos A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad AL &= AG + OG = c \sin B + c \cos B \cot C \\
 &= \frac{c \cos (B-C)}{\sin C} = \frac{a \cos (B-C)}{\sin A};
 \end{aligned}$$

$$\begin{aligned}
 \therefore AL \sin A + BM \sin B + CN \sin C &= a \cos (B-C) + b \cos (C-A) + c \cos (A-B) \\
 &= 2R \{ \sin \overline{B+C} \cos \overline{B-C} + \dots + \dots \} \\
 &= R [(\sin 2B + \sin 2C) + \dots + \dots] \\
 &= 2R [\sin 2A + \sin 2B + \sin 2C] \\
 &= 8R \sin A \sin B \sin C.
 \end{aligned}$$

30. Let P be the centre of the circle inscribed between the in-circle and the sides AB, AC ; then

$$\begin{aligned}
 (1) \quad \frac{r - r_a}{r + r_a} &= \sin \frac{A}{2}; \\
 \therefore r_a &= r \cdot \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}} = r \tan^2 \frac{\pi - A}{4}. \quad [\text{Compare XI, f. Ex. 15.}]
 \end{aligned}$$

$$(1) \quad \text{Also} \quad \frac{\pi - A}{4} + \frac{\pi - B}{4} + \frac{\pi - C}{4} = \frac{\pi}{2};$$

$$\begin{aligned}
 \therefore \sqrt{r_b r_c} + \sqrt{r_c r_a} + \sqrt{r_a r_b} &= r \left(\tan \frac{\pi - B}{4} \tan \frac{\pi - C}{4} + \dots + \dots \right) \\
 &= r.
 \end{aligned}$$

31. It is easily seen that the triangle XYZ is the same as the triangle PQR in Ex. 27.

$$\begin{aligned}\therefore \text{Perimeter} &= \frac{a}{2 \cos B \cos C} + \frac{b}{2 \cos C \cos A} + \frac{c}{2 \cos A \cos B} \\ &= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2 \cos A \cos B \cos C} = 2R \tan A \tan B \tan C.\end{aligned}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \cdot \frac{a}{2 \cos B \cos C} \cdot \frac{b}{2 \cos C \cos A} \cdot \sin 2C \\ &= \frac{ab \sin C}{4 \cos A \cos B \cos C} = R^2 \tan A \tan B \tan C.\end{aligned}$$

32. Let XYZ be the triangle formed by the tangents, and let the perpendiculars from X, Y, Z to the chord be represented by x, y, z respectively.

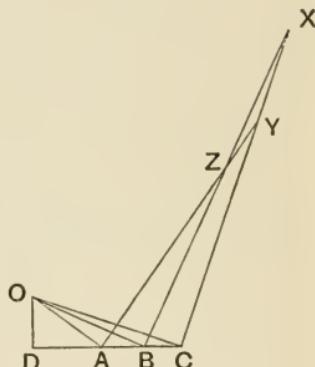
Then the area of

$$\triangle XYZ = \triangle BXC - \triangle ACY + \triangle AZB$$

$$= \frac{1}{2}x \cdot BC - \frac{1}{2}y \cdot AC + \frac{1}{2}z \cdot AB.$$

Now $\frac{x}{CX} = \frac{DC}{CO}$, since OCX is a right angle.

Also $\frac{CX}{CO} = \frac{DB}{OD} = \frac{DB}{p}$, since O, B, C, X are concyclic.



$$\therefore x = \frac{DC \cdot DB}{p}.$$

$\therefore \triangle BXC = \frac{1}{2p} \cdot DC \cdot DB (DC - DB) = \frac{cb(c-b)}{2p}$, if a, b, c denote the distances of A, B, C from D .

$$\begin{aligned}\therefore \text{area of } \triangle XYZ &= \frac{cb(c-b) - ca(c-a) + ba(b-a)}{2p} \\ &= \frac{cb(c-b) + ba(b-a) + ac(a-c)}{2p} \\ &= -\frac{(c-b)(b-a)(a-c)}{2p} = \frac{BC \cdot CA \cdot AB}{2p},\end{aligned}$$

for $CA = c-a = -(a-c)$.

MISCELLANEOUS EXAMPLES. F. PAGE 228.

1. We have $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$;

$$\therefore \cos \alpha \cos \beta - \sin \alpha \sin \beta = \sin \gamma \sin \delta - \cos \gamma \cos \delta;$$

$$\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta.$$

2. First side = $\frac{\cos(15^\circ - A) \sin 15^\circ - \sin(15^\circ - A) \cos 15^\circ}{\sin 15^\circ \cos 15^\circ}$

$$= \frac{2 \sin A}{\sin 30^\circ} = 4 \sin A.$$

3. $\cot A + \sin A \operatorname{cosec} B \operatorname{cosec} C = \frac{\cos A \sin B \sin C + 1 - \cos^2 A}{\sin A \sin B \sin C}$
 $= \frac{\cos A \{ \sin B \sin C + \cos(B+C) \} + 1}{\sin A \sin B \sin C}$
 $= \frac{\cos A \cos B \cos C + 1}{\sin A \sin B \sin C},$

which is symmetrical with respect to A, B, C .

4. We have $\sin B = \frac{b \sin A}{a} = \frac{\sqrt{8} \cdot \sin 30^\circ}{2} = \frac{1}{\sqrt{2}}$;

$$\therefore B = 45^\circ, \text{ or } 135^\circ; \quad [\text{Art. 148. (iii)}];$$

$$\therefore C = 105^\circ, \text{ or } 15^\circ;$$

$$\therefore c = \frac{a \sin C}{\sin A} = 4 \cos 15^\circ, \text{ or } 4 \sin 15^\circ;$$

$$\therefore c = \sqrt{6} + \sqrt{2}, \text{ or } \sqrt{6} - \sqrt{2}.$$

5. (1) We have $\cot 18^\circ \tan 36^\circ = \frac{2 \cos 18^\circ \sin 36^\circ}{2 \sin 18^\circ \cos 36^\circ}$
 $= \frac{\sin 54^\circ + \sin 18^\circ}{\sin 54^\circ - \sin 18^\circ}$
 $= \frac{\sqrt{5} + 1 + (\sqrt{5} - 1)}{\sqrt{5} + 1 - (\sqrt{5} - 1)} = \sqrt{5}.$

(2) $\sin 36^\circ = \sin 144^\circ$, and $\sin 72^\circ = \sin 108^\circ$;

$$\therefore \text{first side} = 4 (2 \sin 72^\circ \sin 36^\circ)^2 = 4 (\cos 36^\circ - \cos 108^\circ)^2$$

$$= 4 (\cos 36^\circ + \sin 18^\circ)^2 = 4 \left(\frac{\sqrt{5} + 1}{4} + \frac{\sqrt{5} - 1}{4} \right)^2 \\ = 5.$$

6. $\log 2 = .30103; \therefore \log 4 = .60206$

$\log 3 = .47712; \therefore \log 9 = .95424$

$$\begin{array}{r} \log 11 = 1.04139 \\ \hline 2.59769 \end{array}$$

$\therefore \log .0396 = -2.59769;$

$\therefore \log (.0396)^{90} = -180 + 53.7921 = \overline{127.7921};$

\therefore number of ciphers before the first significant digit in $(.0396)^{90}$ is 126.

7. Let P, Q be the two positions of the observer;

then $\angle QPB = 30^\circ, \angle QBP = 45^\circ, PQ = 50$ yards;

$$\therefore PB = \frac{50 \sin 75^\circ}{\sin 45^\circ} = \frac{50 \sqrt{2} (\sqrt{3} + 1)}{2 \sqrt{2}} = 25 (\sqrt{3} + 1) = 68.3 \text{ yds.},$$

$$QB = \frac{50 \sin 30^\circ}{\sin 45^\circ} = 50 \sqrt{2} \cdot \frac{1}{2} = 35.35 \text{ yds.}$$

8. First side $= 2 + \frac{1}{2} \{ \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2(\alpha + \beta + \gamma) \}$

$$= 2 + \cos(\alpha + \beta) \cos(\alpha - \beta) + \cos(\alpha + \beta + 2\gamma) \cos(\alpha + \beta)$$

$$= 2 + 2 \cos(\alpha + \beta) \cos(\beta + \gamma) \cos(\gamma + \alpha).$$

9. (1) $\tan 40^\circ + \cot 40^\circ = \frac{\sin^2 40^\circ + \cos^2 40^\circ}{\cos 40^\circ \sin 40^\circ} = \frac{2}{2 \sin 40^\circ \cos 40^\circ}$

$$= \frac{2}{\sin 80^\circ} = 2 \sec 10^\circ.$$

(2) $\tan 70^\circ + \tan 20^\circ = \tan 20^\circ + \cot 20^\circ = \frac{2}{\sin 40^\circ}$, as in (1),
 $= 2 \operatorname{cosec} 40^\circ.$

10. (1) First side $= 2 \sin 4\alpha - 2 \cos 6\alpha \sin 4\alpha$

$$= 2 \sin 4\alpha (1 - \cos 6\alpha) = 4 \sin 4\alpha \sin^2 3\alpha$$

$$= 16 \sin \alpha \cos \alpha \cos 2\alpha \sin^2 3\alpha.$$

(2) First side $= \sin \frac{2\pi}{7} - 2 \cos \frac{5\pi}{7} \sin \frac{\pi}{7}$

$$= 2 \sin \frac{\pi}{7} \left(\cos \frac{\pi}{7} - \cos \frac{5\pi}{7} \right)$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{5\pi}{7} \sin \frac{3\pi}{7}.$$

11. We have $\sin C = \frac{c \sin B}{b} = \frac{6 - 2\sqrt{3}}{2(3\sqrt{2} - \sqrt{6})} = \frac{1}{\sqrt{2}}$;
 $\therefore C = 45^\circ$ or 135° ; [Art. 148, (iii)];

$$\therefore A = 105^\circ \text{ or } 15^\circ;$$

$$\therefore a = \frac{c \sin A}{\sin C} = \frac{(6 - 2\sqrt{3})\sqrt{2}(\sqrt{3} + 1)}{2\sqrt{2}}, \text{ or } \frac{(6 - 2\sqrt{3})\sqrt{2}(\sqrt{3} - 1)}{2\sqrt{2}};$$

that is, $a = 2\sqrt{3}$, or $4\sqrt{3} - 6$.

12. Let C be the rock and A, B the two positions of the ship.

Then we have $\angle BAC = \angle BCA = 67\frac{1}{2}^\circ$;

$$\therefore BC = BA = 10 \text{ miles},$$

and $AC = 2AB \sin 22\frac{1}{2}^\circ = 10\sqrt{2 - \sqrt{2}}$ miles. [Art. 251.]

13. First side $= \frac{\sin^2 B - \sin^2 C}{\cos B + \cos C} + \dots + \dots$
 $= \frac{\cos^2 C - \cos^2 B}{\cos B + \cos C} + \dots + \dots$
 $= (\cos C - \cos B) + \dots + \dots$
 $= 0.$

14. We have $\frac{1}{\cos(\theta - \alpha)} + \frac{1}{\cos(\theta + \alpha)} = \frac{2}{\cos \theta}$;
 $\therefore \frac{4 \cos \theta \cos \alpha}{\cos 2\theta + \cos 2\alpha} = \frac{2}{\cos \theta}$;
 $\therefore 2 \cos^2 \theta \cos \alpha = 2 \cos^2 \theta - 1 + 2 \cos^2 \alpha - 1$;
 $\therefore \cos^2 \theta (\cos \alpha - 1) = \cos^2 \alpha - 1$;
 $\therefore \cos^2 \theta = \cos \alpha + 1 = 2 \cos^2 \frac{\alpha}{2}$;

whence $\cos \theta = \sqrt{2} \cos \frac{\alpha}{2}$.

15. We have $\sin \alpha \cos \alpha = \sin^2 \beta$
 $= \frac{1}{2}(1 - \cos 2\beta)$;
 $\therefore \cos 2\beta = 1 - 2 \sin \alpha \cos \alpha = (\cos \alpha - \sin \alpha)^2$
 $= 2 \cos^2 \left(\frac{\pi}{4} + \alpha \right)$.

16. See figure of Art. 223.

$$OG = BG \cot BOG = BG \cot C = c \cos B \cot C$$

$$= 2R \cos B \cos C;$$

similarly $OH = 2R \cos C \cos A$, $OK = 2R \cos A \cos B$.

17. We have $\cos \theta = \frac{\cos u - e}{1 - e \cos u}$.

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - e \cos u - \cos u + e}{1 - e \cos u + \cos u - e} = \frac{(1 + e)(1 - \cos u)}{(1 - e)(1 + \cos u)};$$

$$\therefore \tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{u}{2}.$$

$$\begin{aligned} 18. \quad & \text{The sum of the squares on the sides containing the right angle} \\ &= 4(1 + \sin \theta)^2 + 4(1 + \cos \theta)^2 + \cos^2 \theta + \sin^2 \theta + 4 \cos \theta (1 + \sin \theta) + 4 \sin \theta (1 + \cos \theta) \\ &= 8 + 8(\sin \theta + \cos \theta) + 4 + 1 + 4(\sin \theta + \cos \theta) + 8 \sin \theta \cos \theta \\ &= 9 + 12(\sin \theta + \cos \theta) + 4(1 + 2 \sin \theta \cos \theta) \\ &= 9 + 12(\sin \theta + \cos \theta) + 4(\sin \theta + \cos \theta)^2 \\ &= \{3 + 2(\sin \theta + \cos \theta)\}^2. \end{aligned}$$

$$\therefore \text{hypotenuse} = 3 + 2(\sin \theta + \cos \theta).$$

19. In the figure of Art. 227 let O be the centre of in-circle of $I_1 I_2 I_3$, and O_1 the centre of the ex-circle opposite to I_1 . Let R' be the circum-radius of $I_1 I_2 I_3$.

Then, as in Art. 220, $OO_1 = 4R' \sin \frac{I_2 I_1 I_3}{2}$;

$$\text{but } \angle I_2 I_1 I_3 = \frac{\pi}{2} - \frac{A}{2}; \text{ and } R' = 2R; \quad [\text{Arts. 221, 222}];$$

$$\therefore OO_1 = 8R \sin \left(\frac{\pi}{4} - \frac{A}{4} \right) = 8R \sin \frac{B+C}{4}.$$

20. The sides of the ex-central triangle of the triangle $I_1 I_2 I_3$ are

$$4R' \cos \frac{\pi - A}{4}, \quad 4R' \cos \frac{\pi - B}{4}, \quad 4R' \cos \frac{\pi - C}{4}, \quad [\text{Art. 221}],$$

$$\text{that is, } 8R \cos \frac{B+C}{4}, \quad 8R \cos \frac{C+A}{4}, \quad 8R \cos \frac{A+B}{4}.$$

21. We have

$$(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \gamma) = (1 - \cos \alpha)(1 - \cos \beta)(1 - \cos \gamma);$$

$$\therefore \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \pm \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2};$$

$$\therefore \text{each expression} = 8 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}$$

$$= \pm 8 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cdot \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$= \pm \sin \alpha \sin \beta \sin \gamma.$$

22. Let $\alpha, \beta, \gamma, \delta$ be four angles such that $\alpha + \beta + \gamma + \delta = 180^\circ$;
 then $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$;
 $\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta$;
 similarly $\cos \beta \cos \gamma + \cos \delta \cos \alpha = \sin \beta \sin \gamma + \sin \delta \sin \alpha$;
 $\cos \gamma \cos \alpha + \cos \beta \cos \delta = \sin \gamma \sin \alpha + \sin \beta \sin \delta$;
 \therefore by addition we have the sum of the products of the cosines taken two together equal to the sum of the products of the sines taken two together.

23. (1) $II_1 \cdot II_2 \cdot II_3 = 4R \sin \frac{A}{2} \cdot 4R \sin \frac{B}{2} \cdot 4R \sin \frac{C}{2}$ [Art. 220]

$$= 16R^2 \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 16R^2 r.$$

(2) $II_1^2 + II_3^2 = 16R^2 \sin^2 \frac{A}{2} + 16R^2 \cos^2 \frac{A}{2}$ [Arts. 220, 221]

$$= 16R^2.$$

24. (1) Let α, β, γ be the angles;

then $\cos \alpha = \frac{\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{1 + \cos \frac{B+C}{2} \cos \frac{B-C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$

$$= \frac{\sin \frac{A}{2} \cos \frac{B-C}{2} + \sin^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \sin \frac{A}{2} = \cos \left(90^\circ - \frac{A}{2} \right).$$

Thus the angles are $90^\circ - \frac{A}{2}, 90^\circ - \frac{B}{2}, 90^\circ - \frac{C}{2}$.

(2) Here

$$\cos \alpha = \frac{\sin^2 2B + \sin^2 2C - \sin^2 2A}{2 \sin 2B \sin 2C} = \frac{1 - \cos 2(B+C) \cos 2(B-C) - \sin^2 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \cos 2(B-C) + \cos 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \{\cos 2(B-C) - \cos 2(B+C)\}}{2 \sin 2B \sin 2C} = -\cos 2A.$$

Thus the angles are $180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$.

25. The expression

$$\begin{aligned}
 &= \{\sin(\theta + \alpha) + \sin(\theta + \beta)\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &\quad - 2 \cos(\alpha - \beta) \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &= \left\{ 2 \sin \frac{2\theta + \alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) (1 + \cos \alpha - \beta) \\
 &= 4 \cos^2 \frac{\alpha - \beta}{2} \left\{ \sin^2 \frac{2\theta + \alpha + \beta}{2} - \sin(\theta + \alpha) \sin(\theta + \beta) \right\} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} \{1 - \cos(2\theta + \alpha + \beta) - \cos(\alpha - \beta) + \cos(2\theta + \alpha + \beta)\} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} (1 - \cos \alpha - \beta),
 \end{aligned}$$

which is independent of θ .

26. See figure on page 220.

Since the quadrilateral is described about a circle,

$$\therefore a + c = b + d; \text{ that is, } a - d = b - c.$$

$$\text{Now } a^2 + d^2 - 2ad \cos A = BD^2 = b^2 + c^2 - 2bc \cos C;$$

$$\therefore (a - d)^2 + 2ad(1 - \cos A) = (b - c)^2 + 2bc(1 - \cos C);$$

$$\therefore ad \sin^2 \frac{A}{2} = bc \sin^2 \frac{C}{2}.$$

27. Let the tangent parallel to BC meet AC in M ; and let AG , the perpendicular from A to BC , meet the tangent in X ; then

$$\frac{p}{a} = \frac{AM}{AC} = \frac{AX}{AG} = \frac{AG - 2r}{AG}$$

$$= 1 - \frac{ar}{\Delta} = 1 - \frac{a}{s};$$

$$\therefore \frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 - \frac{a}{s} + 1 - \frac{b}{s} + 1 - \frac{c}{s}$$

$$= 3 - \frac{2s}{s} = 1.$$

28. Take the figure of Art. 199. Then we have

$$\angle PAB = 13^\circ 14' 12'', \quad \angle PBC = 56^\circ 24' 36'', \quad \angle BPA = 43^\circ 10' 24''.$$

Also $PC = 1566$. Let $AB = x$ ft.

$$x = \frac{PB \sin 43^\circ 10' 24''}{\sin 13^\circ 14' 12''}$$

$$\begin{array}{rcl} \log \sin 43^\circ 10' & = & \bar{1}.8351341 \\ \frac{24}{60} \times 1347 & = & 539 \end{array}$$

$$\begin{array}{rcl} \log PB & = & 3.2741376 \\ & & \underline{3.1093256} \end{array}$$

$$\log \sin 13^\circ 14' 12'' = \bar{1}.3597858$$

$$\log x = 3.7495398$$

$$\log 5617.4 = 3.7495353$$

$$\begin{array}{rcl} & & 45 \\ & & \underline{46} \\ 6 & & \end{array}$$

$$PB = 1566 \operatorname{cosec} 56^\circ 24' 36''.$$

$$\log \operatorname{cosec} 56^\circ 24' = .0793961$$

$$\text{subtract } \frac{36}{60} \times 839 = 503$$

$$\log 1566 = \bar{1}.0793458$$

$$\log PB = \bar{1}.2741376$$

$$\log \sin 13^\circ 14' = \bar{1}.3596785$$

$$\begin{array}{rcl} & & 12 \\ & & \underline{60} \\ & & 1073 \\ & & \end{array}$$

$$\bar{1}.3597858$$

Thus $x = 5617.46$ ft., whence it easily follows that the speed of the train is 21.3 miles per hour.

29. Let A represent the harbour, C the fort, B the position of the ship when 20 miles from C .

Then $AC = 27.23$ miles, $CB = 20$ miles, $\angle CAB = 46^\circ 8' 8.6''$.

$$\sin B = \frac{27.23 \sin 46^\circ 8' 8.6''}{20}.$$

$$\log 27.23 = 1.4350476$$

$$\log \sin 46^\circ 8' = \bar{1}.8579078$$

$$\begin{array}{rcl} \frac{86}{600} \times 1215 & = & 174 \\ & & \underline{1.2929728} \end{array}$$

$$\log 20 = 1.3010300$$

$$\log \sin B = \bar{1}.9919428$$

$$\log \sin 78^\circ 59' = \bar{1}.9919220$$

$$\begin{array}{l} \text{Diff. for } 60'' = 246; \\ \therefore \text{prop'l. increase} = \frac{208}{246} \times 60'' = 50.7''. \end{array}$$

$\therefore B = 78^\circ 59' 50.7''$, or $101^\circ 0' 9.3''$, both values being admissible since $a < b$.

Hence with the third figure of page 131 we have

$$\angle ACB_1 = 54^\circ 52' 0.7'', \quad \angle ACB_2 = 32^\circ 51' 42.1''.$$

In $\triangle ACB_1$,

$$\begin{aligned}AB_1 &= \frac{20 \sin 54^\circ 52' 0.7''}{\sin 46^\circ 8' 8.6''}, \\ \log \sin 54^\circ 52' &= \bar{1}.9126551 \\ \frac{7}{600} \times 889 &= 10 \\ \log 20 &= 1.3010300 \\ &\quad \underline{1.2136861} \\ \log \sin 46^\circ 8' 8.6'' &= \bar{1}.8579252 \\ \log AB_1 &= \bar{1}.3557609 \\ \log 22.686 &= \bar{1}.3557579 \\ &\quad \underline{30} \\ 2 &= \underline{\underline{38}}\end{aligned}$$

$$\therefore AB_1 = 22.6862 \text{ miles.}$$

In $\triangle ACB_2$,

$$\begin{aligned}AB_2 &= \frac{20 \sin 32^\circ 51' 42.1''}{\sin 46^\circ 8' 8.6''}; \\ \log \sin 32^\circ 51' &= \bar{1}.7343529 \\ \frac{421}{600} \times 1956 &= 1372 \\ \log 20 &= 1.3010300 \\ &\quad \underline{1.0355201} \\ \log \sin 46^\circ 8' 8.6'' &= \bar{1}.8579252 \\ \log AB_2 &= \bar{1}.1775949 \\ \log 15.052 &= \bar{1}.1775942\end{aligned}$$

$$\therefore AB_2 = 15.052 \text{ miles.}$$

Thus the time taken is approximately 2.27 hours or 1.5 hours; that is the ship will be 20 miles from the fort in 2 hrs. 16 min. or in 1 hr. 30 min.

EXAMPLES. XIX. a. PAGE 235.

$$1. \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}.$$

$$2. \sin \theta = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4}.$$

$$3. \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

$$4. \tan \theta = \sqrt{3} = \tan \frac{\pi}{3};$$

$$\therefore \theta = n\pi + \frac{\pi}{3}.$$

$$5. \cot \theta = -\sqrt{3} = \cot \left(-\frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi - \frac{\pi}{6}.$$

$$6. \sec \theta = -\sqrt{2} = \sec \frac{3\pi}{4};$$

$$\therefore \theta = 2n\pi \pm \frac{3\pi}{4}.$$

$$7. \cos^2 \theta = \frac{1}{2};$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{2}};$$

$$\theta = 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{4} \right).$$

$$8. \tan^2 \theta = \frac{1}{3};$$

$$\therefore \tan \theta = \pm \frac{1}{\sqrt{3}} = \tan \left(\pm \frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$

Both of these are included in $n\pi \pm \frac{\pi}{4}$.

9. $\operatorname{cosec}^2 \theta = \frac{4}{3};$

$$\therefore \cot^2 \theta = \frac{1}{3};$$

$$\therefore \cot \theta = \pm \frac{1}{\sqrt{3}} = \cot \left(\pm \frac{\pi}{3} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{3}.$$

10. $\cos \theta = \cos \alpha;$

$$\therefore \theta = 2n\pi \pm \alpha.$$

11. $\tan^2 \theta = \tan^2 \alpha;$

$$\therefore \tan \theta = \pm \tan \alpha = \tan (\pm \alpha);$$

$$\therefore \theta = n\pi \pm \alpha.$$

12. $\sec^2 \theta = \sec^2 \alpha;$

$$\therefore \tan^2 \theta = \tan^2 \alpha;$$

$$\therefore \theta = n\pi \pm \alpha.$$

13. $\tan 2\theta = \tan \theta;$

$$\therefore 2\theta = n\pi + \theta;$$

$$\therefore \theta = n\pi.$$

14. $\operatorname{cosec} 3\theta = \operatorname{cosec} 3\alpha;$

$$\therefore 3\theta = n\pi + (-1)^n 3\alpha;$$

$$\therefore \theta = \frac{n\pi}{3} + (-1)^n \alpha.$$

15. $\cos 3\theta = \cos 2\theta;$

$$\therefore 3\theta = 2n\pi \pm 2\theta;$$

$$\therefore \theta = 2n\pi, \text{ or } \frac{2n\pi}{5}.$$

16. $\sin 5\theta + \sin \theta = \sin 3\theta;$

$$\therefore 2 \sin 3\theta \cos 2\theta = \sin 3\theta;$$

$$\therefore \sin 3\theta = 0,$$

or $\cos 2\theta = \frac{1}{2},$

whence $\theta = \frac{n\pi}{3}, \text{ or } n\pi \pm \frac{\pi}{6}.$

17. $\cos \theta - \cos 7\theta = \sin 4\theta;$

$$\therefore 2 \sin 4\theta \sin 3\theta = \sin 4\theta;$$

$$\therefore \sin 4\theta = 0, \text{ or } \sin 3\theta = \frac{1}{2};$$

whence $\theta = \frac{n\pi}{4}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}.$

18.

$$\sin 4\theta + \sin 2\theta - (\sin 3\theta + \sin \theta) = 0;$$

$$\therefore 2 \sin 3\theta \cos \theta - 2 \sin 2\theta \cos \theta = 0;$$

$$\therefore 2 \cos \theta \cdot 2 \cos \frac{5\theta}{2} \sin \frac{\theta}{2} = 0;$$

$$\therefore \cos \theta = 0, \text{ or } \cos \frac{5\theta}{2} = 0, \text{ or } \sin \frac{\theta}{2} = 0.$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{5}, \text{ or } 2n\pi.$$

19. As in Example 18, we obtain

$$4 \cos \theta \cos 4\theta \cos 2\theta = 0;$$

whence $\theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{4}, \text{ or } \frac{(2n+1)\pi}{8}.$

20.

$$\sin 5\theta \cos \theta = \sin 6\theta \cos 2\theta;$$

$$\therefore \sin 6\theta + \sin 4\theta = \sin 8\theta + \sin 4\theta;$$

$$\therefore \sin 8\theta - \sin 6\theta = 0;$$

$$\therefore 2 \sin \theta \cos 7\theta = 0;$$

$$\therefore \sin \theta = 0, \text{ or } \cos 7\theta = 0;$$

whence

$$\theta = n\pi, \text{ or } \frac{(2n+1)\pi}{14}.$$

21.

$$\sin 11\theta \sin 4\theta + \sin 5\theta \sin 2\theta = 0;$$

$$\therefore \cos 7\theta - \cos 15\theta + \cos 3\theta - \cos 7\theta = 0;$$

$$\therefore 2 \sin 9\theta \sin 6\theta = 0;$$

whence

$$\theta = \frac{n\pi}{9}, \text{ or } \frac{n\pi}{6}.$$

22.

$$\sqrt{2} \cos 3\theta - \cos \theta = \cos 5\theta;$$

$$\therefore \sqrt{2} \cos 3\theta = 2 \cos 3\theta \cos 2\theta;$$

$$\therefore \cos 3\theta = 0, \text{ or } \cos 2\theta = \frac{1}{\sqrt{2}};$$

$$\therefore \theta = \frac{(2n+1)\pi}{6}, \text{ or } n\pi \pm \frac{\pi}{8}.$$

23.

$$\sin 7\theta - \sqrt{3} \cos 4\theta = \sin \theta;$$

$$\therefore 2 \cos 4\theta \sin 3\theta = \sqrt{3} \cos 4\theta;$$

$$\therefore \cos 4\theta = 0, \text{ or } \sin 3\theta = \frac{\sqrt{3}}{2};$$

whence

$$\theta = \frac{(2n+1)\pi}{8}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{9}.$$

24.

$$1 + \cos \theta = 2 \sin^2 \theta;$$

$$\therefore 1 + \cos \theta = 2 - 2 \cos^2 \theta;$$

$$\therefore 2 \cos^2 \theta + \cos \theta - 1 = 0;$$

$$\therefore \cos \theta = -1, \text{ or } \frac{1}{2};$$

$$\therefore \theta = (2n+1)\pi, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

25.

$$\tan^2 \theta + \sec \theta = 1;$$

$$\therefore \sec^2 \theta + \sec \theta - 2 = 0;$$

$$\therefore \sec \theta = -2, \text{ or } 1;$$

$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}, \text{ or } 2n\pi.$$

26.

$$\begin{aligned} \cot^2 \theta - 1 &= \operatorname{cosec} \theta; \\ \therefore \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 2 &= 0; \\ \therefore \operatorname{cosec} \theta &= 2, \text{ or } -1; \\ \therefore \theta &= n\pi + (-1)^n \frac{\pi}{6}, \text{ or } n\pi + (-1)^n \frac{3\pi}{2}. \end{aligned}$$

27.

$$\begin{aligned} \cot \theta - \tan \theta &= 2; \\ \therefore \cot^2 \theta - 1 &= 2 \cot \theta; \\ \therefore \cot 2\theta &= 1; \\ \therefore \theta &= \frac{n\pi}{2} + \frac{\pi}{8}. \end{aligned}$$

28.

$$\begin{aligned} 2 \cos \theta &= -1; \quad \therefore \theta = 2n\pi \pm \frac{2\pi}{3} \quad (1), \\ 2 \sin \theta &= \sqrt{3}; \quad \therefore \theta = n\pi + (-1)^n \frac{2\pi}{3} \quad (2). \end{aligned}$$

From (1) we see that the multiple of π must be even, and from (2) that the sign before the second term must be positive when the multiple of π is even;

$$\therefore \theta = 2n\pi + \frac{2\pi}{3}.$$

29.

$$\begin{aligned} \sec \theta &= \sqrt{2}; \quad \therefore \theta = 2n\pi \pm \frac{\pi}{4} \quad (1), \\ \tan \theta &= -1; \quad \therefore \theta = n\pi - \frac{\pi}{4} \quad (2). \end{aligned}$$

From (1) we see that the multiple of π must be even, and from (2), that the sign before the second term must be negative;

$$\therefore \theta = 2n\pi - \frac{\pi}{4}.$$

EXAMPLES. XIX. b. PAGE 237.

1.

$$\begin{aligned} \tan p\theta &= \cot q\theta; \\ \therefore \tan p\theta &= \tan \left(\frac{\pi}{2} - q\theta \right). \\ \therefore p\theta &= n\pi + \frac{\pi}{2} - q\theta; \\ \therefore \theta &= \frac{(2n+1)\pi}{2(p+q)}. \end{aligned}$$

2.

$$\sin m\theta + \cos n\theta = 0;$$

$$\therefore \cos n\theta = -\cos \left(\frac{\pi}{2} + m\theta \right);$$

$$\therefore n\theta = 2k\pi \pm \left(\frac{\pi}{2} + m\theta \right);$$

$$\therefore \theta = \frac{(4k+1)\pi}{2(n-m)}, \text{ or } \frac{(4k-1)\pi}{2(n+m)}.$$

3.

$$\cos \theta - \sqrt{3} \sin \theta = 1;$$

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2};$$

$$\therefore \cos \left(\theta + \frac{\pi}{3} \right) = \cos \frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi, \text{ or } 2n\pi - \frac{2\pi}{3}.$$

4.

$$\sin \theta - \sqrt{3} \cos \theta = 1;$$

$$\therefore \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta = \frac{1}{2};$$

$$\therefore \cos \left(\theta + \frac{\pi}{6} \right) = \cos \frac{2\pi}{3};$$

$$\therefore \theta + \frac{\pi}{6} = 2n\pi \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi - \frac{5\pi}{6};$$

that is,

$$\theta = 2n\pi + \frac{\pi}{2}, \text{ or } (2n+1)\pi + \frac{\pi}{6}.$$

5.

$$\cos \theta = \sqrt{3} (1 - \sin \theta);$$

$$\therefore \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = \frac{\sqrt{3}}{2};$$

$$\therefore \cos \left(\theta - \frac{\pi}{3} \right) = \cos \frac{\pi}{6};$$

$$\therefore \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi + \frac{\pi}{6}.$$

6. $\sin \theta + \sqrt{3} \cos \theta = \sqrt{2};$

$$\therefore \cos\left(\theta - \frac{\pi}{6}\right) = \cos \frac{\pi}{4};$$

$$\therefore \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4};$$

$$\therefore \theta = 2n\pi + \frac{5\pi}{12}, \text{ or } 2n\pi - \frac{\pi}{12}.$$

7. $\cos \theta - \sin \theta = \frac{1}{\sqrt{2}};$

$$\therefore \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2},$$

$$\therefore \cos\left(\theta + \frac{\pi}{4}\right) = \cos \frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{12}, \text{ or } 2n\pi - \frac{7\pi}{12}.$$

8. $\cos \theta + \sin \theta + \sqrt{2} = 0;$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = -1;$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \pi;$$

$$\therefore \theta = 2n\pi + \frac{5\pi}{4}, \text{ or } 2n\pi - \frac{3\pi}{4}.$$

9. $\operatorname{cosec} \theta + \cot \theta = \sqrt{3};$

$$\therefore 1 + \cos \theta = \sqrt{3} \sin \theta;$$

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = -\frac{1}{2};$$

$$\therefore \cos\left(\theta + \frac{\pi}{3}\right) = -\frac{1}{2};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3};$$

whence $\theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n-1)\pi;$

which may be written $\theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n+1)\pi.$

10. $\cot \theta - \cot 2\theta = 2;$

$$\therefore \operatorname{cosec} 2\theta = 2;$$

$$\therefore \theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.$$

11. $2 \sin \theta \sin 3\theta = 1;$

$$\therefore \cos 2\theta - \cos 4\theta = 1;$$

$$\therefore \cos 2\theta = 2 \cos^2 2\theta;$$

$$\therefore \cos 2\theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{4}, \text{ or } n\pi \pm \frac{\pi}{6}.$$

12. $\sin 3\theta = 8 \sin^3 \theta;$

$$\therefore 3 \sin \theta - 4 \sin^3 \theta = 8 \sin^3 \theta;$$

$$\therefore \sin \theta = 4 \sin^3 \theta;$$

$$\therefore \sin \theta = 0, \text{ or } \pm \frac{1}{2};$$

$$\therefore \theta = n\pi, \text{ or } n\pi \pm \frac{\pi}{6}.$$

13. $\tan \theta + \tan 3\theta = 2 \tan 2\theta;$

$$\therefore \frac{\sin 4\theta}{\cos 3\theta \cos \theta} = \frac{2 \sin 2\theta}{\cos 2\theta};$$

$$\therefore \sin 2\theta = 0; \text{ whence } \theta = \frac{n\pi}{2},$$

or $\cos^2 2\theta = \cos \theta \cos 3\theta;$

$$\therefore 2 \cos^2 2\theta = \cos 4\theta + \cos 2\theta = 2 \cos^2 2\theta - 1 + \cos 2\theta;$$

$$\therefore \cos 2\theta = 1;$$

whence $\theta = n\pi;$

\therefore all the values are included in $\theta = \frac{n\pi}{2}.$

14. $\cos \theta - \sin \theta = \cos 2\theta;$

$$\therefore 2 \sin \frac{3\theta}{2} \sin \frac{\theta}{2} = \sin \theta;$$

$$\therefore \sin \frac{\theta}{2} = 0, \text{ whence } \theta = 2n\pi,$$

or $\sin \frac{3\theta}{2} = \cos \frac{\theta}{2};$

$$\therefore \frac{\theta}{2} = 2n\pi \pm \left(\frac{\pi}{2} - \frac{3\theta}{2} \right);$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } -\theta = 2n\pi - \frac{\pi}{2};$$

\therefore the values of θ may be written $2n\pi, n\pi + \frac{\pi}{4}, 2n\pi + \frac{\pi}{2}.$

15.

$$\text{cosec } \theta + \sec \theta = 2\sqrt{2};$$

$$\therefore \sin \theta + \cos \theta = 2\sqrt{2} \sin \theta \cos \theta;$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = 2 \sin \theta \cos \theta = \sin 2\theta;$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right);$$

whence $\theta = \frac{2n\pi}{3} + \frac{\pi}{4}$, or $2n\pi + \frac{\pi}{4}$.

16.

$$\sec \theta - \text{cosec } \theta = 2\sqrt{2};$$

$$\therefore 1 - \sin 2\theta = 8 \sin^2 \theta \cos^2 \theta = 2 \sin^2 2\theta;$$

$$\therefore \sin 2\theta = -1, \text{ or } \frac{1}{2};$$

$$\therefore \theta = n\pi - \frac{\pi}{4}, \text{ or } \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.$$

The equation may also be solved in the same way as Ex. 15.

17.

$$\frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2;$$

$$\cos 2\theta - \cos 4\theta = 2 \cos 4\theta \cos 2\theta \\ = \cos 6\theta + \cos 2\theta;$$

$$\therefore \cos 6\theta + \cos 4\theta = 0,$$

$$2 \cos 5\theta \cos \theta = 0;$$

$$\therefore \theta = (2n+1) \frac{\pi}{2}, \text{ or } 5\theta = (2n+1) \frac{\pi}{2}.$$

18.

$$\cos 3\theta + 8 \cos^3 \theta = 0;$$

$$\therefore 4 \cos^3 \theta = \cos \theta;$$

$$\therefore \cos \theta = 0, \text{ or } \pm \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \left(\pi - \frac{\pi}{3}\right);$$

that is, the values of θ are $\frac{(2n+1)\pi}{2}, n\pi \pm \frac{\pi}{3}$.

19.

$$1 + \sqrt{3} \tan^2 \theta = (1 + \sqrt{3}) \tan \theta;$$

$$\therefore (\sqrt{3} \tan \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } \frac{1}{\sqrt{3}};$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } n\pi + \frac{\pi}{6}.$$

20.

$$\begin{aligned} \tan^3 \theta + \cot^3 \theta &= 8 \operatorname{cosec}^3 2\theta + 12; \\ \therefore \sin^6 \theta + \cos^6 \theta &= 1 + 12 \sin^3 \theta \cos^3 \theta; \\ \therefore \sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta &= 1 + 12 \sin^3 \theta \cos^3 \theta; \\ \therefore 1 - 3 \sin^2 \theta \cos^2 \theta &= 1 + 12 \sin^3 \theta \cos^3 \theta; \\ \therefore \sin^2 \theta \cos^2 \theta (4 \sin \theta \cos \theta + 1) &= 0; \\ \text{whence } \sin 2\theta &= 0, \text{ or } -\frac{1}{2}; \\ \therefore \theta &= \frac{n\pi}{2}, \text{ or } \frac{n\pi}{2} + (-1)^{n+1} \frac{\pi}{12}. \end{aligned}$$

21.

$$\begin{aligned} \sin \theta &= \sqrt{2} \sin \phi, \quad \sqrt{3} \cos \theta = \sqrt{2} \cos \phi; \\ \therefore \text{by squaring and adding we have } \sin^2 \theta + 3 \cos^2 \theta &= 2, \end{aligned}$$

that is,

$$\begin{aligned} 1 + 2 \cos^2 \theta &= 2, \text{ whence } \cos \theta = \pm \frac{1}{\sqrt{2}}; \\ \therefore \theta &= 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{4} \right); \end{aligned}$$

both of which are included in $\theta = n\pi \pm \frac{\pi}{4}$.

$$\begin{aligned} \text{Again, we have } \cos \phi &= \frac{\sqrt{3}}{\sqrt{2}} \cos \theta = \pm \frac{\sqrt{3}}{2}; \\ \therefore \phi &= 2n\pi \pm \frac{\pi}{6}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{6} \right), \end{aligned}$$

which are both included in $\phi = n\pi \pm \frac{\pi}{6}$.

22. $\operatorname{cosec} \theta = \sqrt{3} \operatorname{cosec} \phi, \cot \theta = 3 \cot \phi.$

By squaring and subtracting we have

$$\begin{aligned} 1 &= 3 (\operatorname{cosec}^2 \phi - 3 \cot^2 \phi) = 3 (1 - 2 \cot^2 \phi); \\ \therefore \cot \phi &= \pm \frac{1}{\sqrt{3}}; \text{ whence } \phi = n\pi \pm \frac{\pi}{3}. \end{aligned}$$

Also

$$\cot \theta = \pm \sqrt{3}; \text{ whence } \theta = n\pi \pm \frac{\pi}{6}.$$

23.

$$\sec \phi = \sqrt{2} \sec \theta, \tan \phi = \sqrt{3} \tan \theta.$$

Subtracting the square of the second equation from the square of the first, we obtain

$$1 = 2 \sec^2 \theta - 3 \tan^2 \theta = 2 - \tan^2 \theta;$$

$$\therefore \tan \theta = \pm 1; \text{ whence } \theta = n\pi \pm \frac{\pi}{4}.$$

Also

$$\tan \phi = \pm \sqrt{3}; \text{ whence } \phi = n\pi \pm \frac{\pi}{3}.$$

24. If

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6};$$

then

$$\sin \left(\theta + \frac{\pi}{4} \right) = \sin \frac{\pi}{6};$$

$$\therefore \cos \left(\frac{\pi}{2} - \theta - \frac{\pi}{4} \right) = \cos \left(\frac{\pi}{2} - \frac{\pi}{6} \right);$$

$$\therefore \cos \left(\theta - \frac{\pi}{4} \right) = \cos \frac{\pi}{3};$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

Which proves that the same series of angles are given by the two equations.

25. In the figure on page 252 let OE be drawn bisecting the angle P_1OP_3 , and let the angles P_1OE , P_2OE each be equal to α ;

then the formula $\left(2n + \frac{1}{4} \right) \pi \pm \alpha$ comprises angles whose boundary lines are OP_1 or OP_3 .

Again in the formula $\left(n - \frac{1}{4} \right) \pi + (-1)^n \left(\frac{\pi}{2} - \alpha \right)$ the terms $n\pi + (-1)^n \frac{\pi}{2}$ comprise angles whose boundary lines are OY , whether n be odd or even.

Hence the whole formula comprises angles formed by starting again from OY and turning through an angle $-\frac{\pi}{4} - (-1)^n \alpha$; that is, $-\frac{\pi}{4} \pm \alpha$.

Hence the second formula also comprises angles whose boundary lines are OP_1 or OP_3 .

EXAMPLES. XIX. c. PAGE 242.

1. Let $\theta = \sin^{-1} \frac{12}{13}$; then $\operatorname{cosec} \theta = \frac{13}{12}$.

$$\therefore \cot^2 \theta = \frac{169}{144} - 1 = \frac{25}{144};$$

$$\therefore \theta = \cot^{-1} \frac{5}{12}.$$

2. Let $\theta = \operatorname{cosec}^{-1} \frac{17}{8}$; then $\operatorname{cosec} \theta = \frac{17}{8}$, and $\cot \theta = \frac{15}{8}$.

$$\therefore \theta = \tan^{-1} \frac{8}{15}.$$

3. Let $\theta = \tan^{-1} x$; then $\tan \theta = x$;

$$\therefore \sec^2 \theta = 1 + \tan^2 \theta = 1 + x^2,$$

$$\sec (\tan^{-1} x) = \sqrt{1 + x^2}.$$

that is,

$$4. \quad 2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \tan^{-1} \frac{3}{4}.$$

$$5. \quad \tan^{-1} \frac{4}{3} - \tan^{-1} 1 = \tan^{-1} \frac{\frac{4}{3} - 1}{1 + \frac{4}{3}} = \tan^{-1} \frac{1}{7}.$$

$$6. \quad \begin{aligned} \tan^{-1} \frac{2}{11} + \cot^{-1} \frac{24}{7} &= \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} \\ &= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{7}{132}} = \tan^{-1} \frac{1}{2}. \end{aligned}$$

$$7. \quad \cot^{-1} \frac{4}{3} - \cot^{-1} \frac{15}{8} = \cot^{-1} \frac{\frac{4}{3} \cdot \frac{15}{8} + 1}{\frac{15}{8} - \frac{4}{3}} = \cot^{-1} \frac{84}{13}.$$

$$8. \quad \begin{aligned} 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} &= \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{25}} + \tan^{-1} \frac{1}{4} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{1}{4} \\ &= \tan^{-1} \frac{\frac{5}{12} + \frac{1}{4}}{1 - \frac{5}{48}} = \tan^{-1} \frac{32}{43}. \end{aligned}$$

$$9. \quad \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \tan^{-1} 1.$$

Again, $\tan^{-1} \frac{5}{6} + \tan^{-1} \frac{1}{11} = \tan^{-1} \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{66}} = \tan^{-1} 1.$

$$\begin{aligned}
 10. \quad & \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} = \tan^{-1} \frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{56}} + \tan^{-1} \frac{1}{18} \\
 & = \tan^{-1} \frac{3}{11} + \tan^{-1} \frac{1}{18} = \tan^{-1} \frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{198}} \\
 & = \tan^{-1} \frac{1}{3} = \cot^{-1} 3.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{5} + \tan^{-1} \frac{3}{4} \\
 & = \tan^{-1} \frac{\frac{3}{5} + \frac{3}{4}}{1 - \frac{9}{20}} = \tan^{-1} \frac{27}{11}.
 \end{aligned}$$

$$12. \quad 2 \cot^{-1} \frac{5}{4} = 2 \tan^{-1} \frac{4}{5} = \tan^{-1} \frac{\frac{8}{5}}{1 - \frac{16}{25}} = \tan^{-1} \frac{40}{9}.$$

$$\begin{aligned}
 13. \quad & 2 \tan^{-1} \frac{8}{15} = \tan^{-1} \frac{\frac{16}{15}}{1 - \frac{64}{225}} = \tan^{-1} \frac{240}{161} = \sin^{-1} \frac{240}{\sqrt{161^2 + 240^2}} \\
 & = \sin^{-1} \frac{240}{289}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \text{Let } \sin^{-1} x = \theta; \text{ then } \sin \theta = x, \cos \theta = \sqrt{1 - x^2}. \\
 & \therefore \sin(2 \sin^{-1} x) = \sin 2\theta = 2 \sin \theta \cos \theta = 2x \sqrt{1 - x^2}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \text{Let } \sin^{-1} \sqrt{\frac{1-x}{2}} = \theta; \text{ then } \sin \theta = \sqrt{\frac{1-x}{2}}. \\
 \text{Now} \quad & \cos 2\theta = 1 - (1-x) = x; \text{ whence } 2\theta = \cos^{-1} x.
 \end{aligned}$$

$$\text{That is, } \cos^{-1} x = 2 \sin^{-1} \sqrt{\frac{1-x}{2}}.$$

16. Let $\tan^{-1} \sqrt{\frac{x}{a}} = \theta$; then $\tan \theta = \sqrt{\frac{x}{a}}$.

Now $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{a - x}{a + x}$; whence $2\theta = \cos^{-1} \frac{a - x}{a + x}$.

That is, $2 \tan^{-1} \sqrt{\frac{x}{a}} = \cos^{-1} \frac{a - x}{a + x}$.

$$\begin{aligned} 17. \quad & 2 \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{1}{8} + \frac{1}{5}}{1 - \frac{1}{40}} + \tan^{-1} \frac{1}{7} \\ & = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} + \tan^{-1} \frac{1}{7} \\ & = \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}} = \tan^{-1} 1 = \frac{\pi}{4}. \end{aligned}$$

18. Let $\theta = \sin^{-1} a$, $\phi = \cos^{-1} b$;

then $\sin \theta = a$, $\cos \theta = \sqrt{1 - a^2}$, $\cos \phi = b$, $\sin \phi = \sqrt{1 - b^2}$;

$$\therefore \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi = b \sqrt{1 - a^2} + a \sqrt{1 - b^2};$$

$$\therefore \sin^{-1} a - \cos^{-1} b = \theta - \phi = \cos^{-1} \{b \sqrt{1 - a^2} + a \sqrt{1 - b^2}\}.$$

[In some of the examples which follow diagrams may be used with advantage as in Examples 2 and 3 of Art. 249.]

$$19. \quad \sin^{-1} \frac{4}{5} + \cos^{-1} \frac{2}{\sqrt{5}} = \cot^{-1} \frac{3}{4} + \cot^{-1} 2 = \cot^{-1} \frac{\frac{3}{2} - 1}{\frac{3}{4} + 2} = \cot^{-1} \frac{2}{11}.$$

$$\begin{aligned} 20. \quad & \cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} = \cos^{-1} \frac{63}{65} + \tan^{-1} \frac{5}{12} \\ & = \tan^{-1} \frac{\sqrt{65^2 - 63^2}}{63} + \tan^{-1} \frac{5}{12} \\ & = \tan^{-1} \frac{16}{63} + \tan^{-1} \frac{5}{12} \\ & = \tan^{-1} \frac{192 + 315}{756 - 80} = \tan^{-1} \frac{507}{676} = \tan^{-1} \frac{3}{4} \\ & = \sin^{-1} \frac{3}{5}. \end{aligned}$$

$$\begin{aligned}
 21. \quad \tan^{-1} m + \tan^{-1} n &= \tan^{-1} \frac{m+n}{1-mn} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(m+n)^2 + (1-mn)^2}} = \cos^{-1} \frac{1-mn}{\sqrt{1+m^2+n^2+m^2n^2}} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.
 \end{aligned}$$

$$22. \text{ Let } \cos^{-1} \frac{20}{29} = \theta, \quad \tan^{-1} \frac{16}{63} = \phi;$$

$$\text{then we have } \sin \theta = \frac{21}{29}, \quad \sin \phi = \frac{16}{63}, \quad \cos \phi = \frac{63}{65};$$

$$\therefore \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi = \frac{20}{29} \cdot \frac{63}{65} + \frac{21}{29} \cdot \frac{16}{65} = \frac{1596}{1885};$$

$$\text{that is, } \cos^{-1} \frac{20}{29} - \tan^{-1} \frac{16}{63} = \theta - \phi = \cos^{-1} \frac{1596}{1885}.$$

$$23. \text{ Let } \cos^{-1} \sqrt{\frac{2}{3}} = \theta, \text{ then } \cos \theta = \sqrt{\frac{2}{3}}, \quad \sin \theta = \frac{1}{\sqrt{3}}.$$

$$\begin{aligned}
 \text{Now } \cos\left(\theta - \frac{\pi}{6}\right) &= \cos \theta \cos \frac{\pi}{6} + \sin \theta \sin \frac{\pi}{6} \\
 &= \sqrt{\frac{2}{3} \cdot \frac{3}{4}} + \sqrt{\frac{1}{3} \cdot \frac{1}{4}} \\
 &= \frac{\sqrt{6}+1}{2\sqrt{3}}.
 \end{aligned}$$

$$\therefore \theta - \frac{\pi}{6} = \cos^{-1} \frac{\sqrt{6}+1}{2\sqrt{3}};$$

$$\text{that is, } \cos^{-1} \sqrt{\frac{2}{3}} - \cos^{-1} \frac{\sqrt{6}+1}{2\sqrt{3}} = \frac{\pi}{6}.$$

$$\begin{aligned}
 24. \quad \text{Second side} &= 2 \cdot \frac{x+x^3}{1-x^4} = \frac{2x}{1-x^2} \\
 &= \tan\left(\tan^{-1} \frac{2x}{1-x^2}\right) \\
 &= \tan(2 \tan^{-1} x).
 \end{aligned}$$

$$25. \quad \text{Second side} = \tan^{-1} a - \tan^{-1} b + \tan^{-1} b - \tan^{-1} c + \tan^{-1} c \\ = \tan^{-1} a.$$

$$26. \text{ Let } \tan^{-1} x = \alpha, \quad \tan^{-1} y = \beta, \quad \tan^{-1} z = \gamma; \\ \text{then } \alpha + \beta + \gamma = \pi;$$

$\therefore \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$; [Art. 135, Ex. 2.]

that is, $x+y+z=xyz$.

27. We have $u = \cot^{-1} \sqrt{\cos \alpha} - \cot^{-1} \sqrt{\frac{1}{\cos \alpha}}$

$$= \cot^{-1} \frac{1+1}{\sqrt{\frac{1}{\cos \alpha}} - \sqrt{\cos \alpha}} = \cot^{-1} \frac{2\sqrt{\cos \alpha}}{1-\cos \alpha};$$

$$\therefore \cot^2 u = \frac{4 \cos \alpha}{(1-\cos \alpha)^2};$$

$$\therefore \operatorname{cosec}^2 u = 1 + \cot^2 u = \left(\frac{1+\cos \alpha}{1-\cos \alpha}\right)^2;$$

$$\therefore \sin u = \frac{1-\cos \alpha}{1+\cos \alpha} = \tan^2 \frac{\alpha}{2}.$$

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1. $\sin^{-1} x = \cos^{-1} x = \sin^{-1} \sqrt{1-x^2};$
 $\therefore x = \sqrt{1-x^2}; \text{ whence } x = \pm \frac{1}{\sqrt{2}}.$

2. $\tan^{-1} x = \cot^{-1} x = \tan^{-1} \frac{1}{x};$
 $\therefore x = \frac{1}{x}; \text{ whence } x = \pm 1.$

3. $\tan^{-1}(x+1) - \tan^{-1}(x-1) = \cot^{-1} 2;$
 $\therefore \tan^{-1} \frac{2}{x^2} = \tan^{-1} \frac{1}{2}; \text{ whence } x = \pm 2.$

4. $\cot^{-1} x + \cot^{-1} 2x = \frac{3\pi}{4};$
 $\therefore \cot^{-1} \frac{2x^2-1}{3x} = \frac{3\pi}{4};$
 $\therefore \frac{2x^2-1}{3x} = \cot \frac{3\pi}{4} = -1;$
 $\therefore 2x^2+3x-1=0; \text{ whence } x = \frac{-3 \pm \sqrt{17}}{4}.$

5. $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2);$
 $\therefore \sin(\sin^{-1} x - \cos^{-1} x) = 3x-2;$
 $\therefore x^2 - \sqrt{1-x^2} \cdot \sqrt{1-x^2} = 3x-2;$

that is,

$$2x^2-1=3x-2;$$

$$\therefore 2x^2-3x+1=0; \text{ whence } x=1, \text{ or } \frac{1}{2}.$$

$$6. \quad \cos^{-1} x - \sin^{-1} x = \cos^{-1} x \sqrt{3}; \\ \therefore \cos(\cos^{-1} x - \sin^{-1} x) = x \sqrt{3}; \\ \therefore x \sqrt{1-x^2} + x \sqrt{1-x^2} = x \sqrt{3};$$

that is, $2x \sqrt{1-x^2} = x \sqrt{3}$; whence $x=0$, or $\pm \frac{1}{2}$.

$$7. \quad \tan^{-1} \frac{x-1}{x+2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}; \\ \therefore \tan^{-1} \frac{\frac{x-1}{x+2} + \frac{x+1}{x+2}}{1 - \frac{x^2-1}{x^2+4}} = \frac{\pi}{4}; \\ \therefore \frac{2x^2+4}{-3} = 1; \\ \therefore 2x^2 = 1, \text{ or } x = \pm \frac{1}{\sqrt{2}}.$$

$$8. \quad 2 \cot^{-1} 2 + \cos^{-1} \frac{3}{5} = \operatorname{cosec}^{-1} x; \\ \therefore \cot^{-1} \frac{3}{4} + \cot^{-1} \frac{3}{4} = \operatorname{cosec}^{-1} x; \\ \therefore \cot^{-1} \frac{\frac{9}{16}-1}{\frac{3}{2}} = \operatorname{cosec}^{-1} x; \\ \therefore \cot^{-1} -\frac{7}{24} = \operatorname{cosec}^{-1} x = \cot^{-1} \sqrt{x^2-1}; \\ \therefore \frac{49}{24^2} = x^2-1; \text{ whence } x = \pm \frac{25}{24}.$$

$$9. \quad \tan^{-1} x + \tan^{-1}(1-x) = 2 \tan^{-1} \sqrt{x-x^2}; \\ \therefore \tan^{-1} \frac{1}{1-x+x^2} = \tan^{-1} \frac{2 \sqrt{x-x^2}}{1-x+x^2}; \\ \therefore 1 = \frac{1}{4}(x-x^2); \text{ whence } x = \frac{1}{2}.$$

$$10. \quad \cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2} = 2 \tan^{-1} x; \\ \therefore \tan^{-1} \frac{2a}{1-a^2} - \tan^{-1} \frac{2b}{1-b^2} = 2 \tan^{-1} x; \\ \therefore 2 \tan^{-1} a - 2 \tan^{-1} b = 2 \tan^{-1} x; \\ \therefore \tan^{-1} a - \tan^{-1} b = \tan^{-1} x; \text{ whence } x = \frac{a-b}{1+ab}.$$

$$11. \quad \sin^{-1} \frac{2a}{1+a^2} + \tan^{-1} \frac{2x}{1-x^2} = \cos^{-1} \frac{1-b^2}{1+b^2};$$

$$\therefore \tan^{-1} \frac{2a}{1-a^2} + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{2b}{1-b^2};$$

$$\therefore 2 \tan^{-1} a + 2 \tan^{-1} x = 2 \tan^{-1} b; \text{ whence } x = \frac{b-a}{1+ab}.$$

$$12. \quad \cot^{-1} \frac{x^2-1}{2x} + \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore 2 \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore \tan^{-1} \frac{2x}{1-x^2} = \frac{2\pi}{3};$$

$$\tan^{-1} x = \frac{\pi}{3}; \text{ whence } x = \sqrt{3}.$$

$$13. \quad \sin^{-1} \frac{2ab}{a^2+b^2} = \sin^{-1} \frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}} = \tan^{-1} \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}};$$

$$\therefore \sin^{-1} \frac{2ab}{a^2+b^2} + \sin^{-1} \frac{2cd}{c^2+d^2} = 2 \tan^{-1} \frac{b}{a} + 2 \tan^{-1} \frac{d}{c}$$

$$= 2 \tan^{-1} \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}}$$

$$= 2 \tan^{-1} \frac{bc+ad}{ac-bd}$$

$$= 2 \tan^{-1} \frac{y}{x} = \sin^{-1} \frac{2xy}{x^2+y^2},$$

where $y = bc + ad, x = ac - bd.$

$$14. \quad \sin [2 \cos^{-1} \{\cot(2 \tan^{-1} x)\}] = 0;$$

$$\therefore \sin \left[2 \cos^{-1} \left\{ \cot \left(\tan^{-1} \frac{2x}{1-x^2} \right) \right\} \right] = 0;$$

$$\therefore \sin \left[2 \cos^{-1} \frac{1-x^2}{2x} \right] = 0;$$

$$\therefore \frac{1-x^2}{2x} \cdot \sqrt{1 - \left(\frac{1-x^2}{2x} \right)^2} = 0;$$

whence $x = \pm 1, \text{ or } 1-x^2 = \pm 2x;$

that is, $x = \pm 1, \text{ or } \pm (1 \pm \sqrt{2}/2).$

15.

$$2 \tan^{-1}(\cos \theta) = \tan^{-1}(2 \operatorname{cosec} \theta);$$

$$\therefore \frac{2 \cos \theta}{1 - \cos^2 \theta} = 2 \operatorname{cosec} \theta;$$

$$\therefore \frac{2 \cos \theta}{\sin^2 \theta} = \frac{2}{\sin \theta};$$

$$\therefore \cot \theta = 1; \text{ whence } \theta = n\pi + \frac{\pi}{4}.$$

16.

$$\sin(\pi \cos \theta) = \cos(\pi \sin \theta);$$

$$\therefore \pi \sin \theta = \frac{\pi}{2} \pm \pi \cos \theta;$$

$$\therefore \sin \theta \pm \cos \theta = \frac{1}{2};$$

$$\therefore 1 \pm 2 \sin \theta \cos \theta = \frac{1}{4};$$

$$\therefore \sin 2\theta = \pm \frac{3}{4};$$

$$\therefore 2\theta = \pm \sin^{-1} \frac{3}{4}.$$

17.

$$\sin(\pi \cot \theta) = \cos(\pi \tan \theta);$$

$$\therefore \pi \tan \theta = 2n\pi \pm \left(\frac{\pi}{2} - \pi \cot \theta \right);$$

$$\therefore \tan \theta \pm \cot \theta = \frac{4n \pm 1}{2};$$

$$\therefore \frac{\sin^2 \theta \pm \cos^2 \theta}{\sin 2\theta} = \frac{4n \pm 1}{4};$$

that is, either $\cot 2\theta$ or $\operatorname{cosec} 2\theta$ is of the form $\frac{4n+1}{4}$.

18.

$$\tan(\pi \cot \theta) = \cot(\pi \tan \theta);$$

$$\therefore \pi \tan \theta = n\pi + \frac{\pi}{2} - \pi \cot \theta;$$

$$\therefore \tan^2 \theta - \frac{2n+1}{2} \tan \theta + 1 = 0;$$

$$\therefore \tan \theta = \frac{2n+1 \pm \sqrt{(2n+1)^2 - 16}}{4}$$

$$= \frac{2n+1}{4} \pm \frac{\sqrt{4n^2 + 4n - 15}}{4}.$$

19.

$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3;$$

$$\therefore \frac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3;$$

$$\therefore y = \frac{3x+1}{3-x};$$

thus we see that the only positive integral values which x may have are 1, 2;

when $x=1, y=2$;

when $x=2, y=7$;

and these are all the positive integral solutions of the equation.

MISCELLANEOUS EXAMPLES. G. PAGE 246.

1. Since

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

we have

$$\frac{a}{4} = \frac{b}{5} = \frac{c}{6};$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{45}{60} = \frac{12}{16}.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{27}{48} = \frac{9}{16}.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5}{40} = \frac{2}{16}.$$

That is, the cosines are in the ratio of 12 : 9 : 2.

2. (1)

$$2 \cos^3 \theta + \sin^2 \theta - 1 = 0;$$

$$\therefore 2 \cos^3 \theta = 1 - \sin^2 \theta = \cos^2 \theta;$$

$$\therefore \cos \theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

(2)

$$\sec^3 \theta - 2 \tan^2 \theta = 2;$$

$$\therefore \sec^3 \theta = 2(1 + \tan^2 \theta) = 2 \sec^2 \theta;$$

$\therefore \sec \theta = 2$, since $\sec \theta = 0$ is inadmissible;

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

3.

$$\tan \beta = 2 \sin \alpha \sin \gamma \operatorname{cosec}(\alpha + \gamma);$$

$$\therefore 2 \cot \beta = \frac{\sin(\alpha + \gamma)}{\sin \alpha \sin \gamma} = \cot \alpha + \cot \gamma;$$

$\therefore \cot \alpha, \cot \beta, \cot \gamma$ are in arithmetical progression.

$$\begin{aligned}
 4. \quad 4r(r_1 + r_2 + r_3) &= \frac{4\Delta^2}{s} \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \\
 &= 4 \{(s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b)\} \\
 &= 4 \{3s^2 - 2(a+b+c)s + bc + ca + ab\} \\
 &= 4(bc + ca + ab) - 4s^2 \\
 &= 4(bc + ca + ab) - (a+b+c)^2 \\
 &= 2(bc + ca + ab) - (a^2 + b^2 + c^2).
 \end{aligned}$$

$$\begin{aligned}
 5. \quad (1) \quad \tan^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} &= \tan^{-1} \frac{\frac{1}{3} - \frac{1}{5}}{1 + \frac{1}{15}} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{\frac{1}{8} + \frac{1}{7}}{1 - \frac{1}{56}} = \tan^{-1} \frac{3}{11}. \\
 (2) \quad \cos \left(\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} \right) &= \frac{4}{5} \cdot \frac{15}{17} - \frac{3}{5} \cdot \frac{8}{17} \\
 &= \frac{36}{85}; \\
 \therefore \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} &= \cos^{-1} \frac{36}{85} \\
 &= \frac{\pi}{2} - \sin^{-1} \frac{36}{85}.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{333 \times 74}{185 \times 222}} = \sqrt{\frac{6}{10}}. \\
 \log \cos \frac{C}{2} &= \frac{1}{2}(\log 6 - 1) = \frac{1}{2}(1.7781513) \\
 &= 1.8890756; \\
 \log \cos 39^\circ 14' &= \frac{1.8890644}{112} \\
 \text{prop'l. decrease} &= \frac{112}{1032} \times 60'' = 6.5''.
 \end{aligned}$$

$\therefore \frac{C}{2} = 39^\circ 13' 53.5'', \text{ and } C = 78^\circ 27' 47''.$

7.

$$\tan(\alpha + \theta) = n \tan(\alpha - \theta);$$

$$\therefore \frac{\tan(\alpha + \theta)}{\tan(\alpha - \theta)} = n;$$

$$\therefore \frac{\tan(\alpha + \theta) - \tan(\alpha - \theta)}{\tan(\alpha + \theta) + \tan(\alpha - \theta)} = \frac{n-1}{n+1};$$

that is,

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{n-1}{n+1}.$$

8.

$$8R^2 = a^2 + b^2 + c^2;$$

$$\therefore 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$= 2 - \cos(A+B) \cos(A-B) - \cos^2 C$$

$$= 2 + 2 \cos A \cos B \cos C;$$

$$\therefore \cos A \cos B \cos C = 0;$$

that is, one of the angles of the triangle is a right angle.

9. Area of inscribed polygon = $nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$;

Area of circumscribed polygon = $ur^2 \tan \frac{\pi}{n}$;

∴ we have

$$\cos^2 \frac{\pi}{n} = \frac{3}{4};$$

$$\therefore \cos \frac{\pi}{n} = \frac{\sqrt{3}}{2};$$

$$\therefore n = 6.$$

10. Let P, Q be the positions of the boats; then we have

$$\angle PBA = \angle QBC = 45^\circ, \quad \angle PCB = 15^\circ, \quad \angle QCD = 75^\circ.$$

$$\therefore PB = \frac{400 \sin 15^\circ}{\sin 30^\circ} = 200\sqrt{2}(\sqrt{3}-1);$$

$$QB = \frac{400 \sin 75^\circ}{\sin 30^\circ} = 200\sqrt{2}(\sqrt{3}+1).$$

But PBQ is a right angle;

$$\therefore PQ^2 = PB^2 + QB^2 = 200^2(12+4) = 200^2 \times 4^2;$$

$$\therefore PQ = 800.$$

Again, the distance of P from $AB = PB \sin 45^\circ = 200(\sqrt{3}-1)$
 $= 146.4$ yds.,and the distance of Q from $AB = QB \sin 75^\circ = 200(\sqrt{3}+1)$
 $= 546.4$ yds.

EXAMPLES. XX. a. PAGE 255.

1. When $\frac{A}{2}$ lies between -135° and -180° , $\sin \frac{A}{2}$ is negative, therefore in the first formula of Art. 254, the negative sign must be taken.

2. $\frac{A}{2}$ lies between 135° and 180° ; and therefore

$$\sin \frac{A}{2} \text{ is positive, } \cos \frac{A}{2} \text{ is negative.}$$

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{169}{256}}{2}} = \sqrt{\frac{87}{256}} = \frac{3\sqrt{119}}{16},$$

$$\cos \frac{A}{2} = -\sqrt{\frac{1 + \cos A}{2}} = -\sqrt{\frac{1 + \frac{169}{256}}{2}} = -\sqrt{\frac{425}{256}} = -\frac{5\sqrt{119}}{16}.$$

3. Here $\sin \frac{A}{2}$ is negative, and $\cos \frac{A}{2}$ is positive;

$$\therefore \sin \frac{A}{2} = -\sqrt{\frac{1 - \cos A}{2}} = -\sqrt{\frac{1 - \frac{289}{256}}{2}} = -\sqrt{\frac{67}{256}} = -\frac{3\sqrt{161}}{16},$$

$$\cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{1 + \frac{289}{256}}{2}} = \sqrt{\frac{545}{256}} = \frac{5\sqrt{161}}{16}.$$

4. When $\frac{A}{2}$ lies between 135° and 225° , $\cos \frac{A}{2} > \sin \frac{A}{2}$ and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A},$$

and

$$\sin \frac{A}{2} - \cos \frac{A}{2} = +\sqrt{1 - \sin A}.$$

7. When $\frac{A}{2}$ lies between 45° and 90° , $\sin \frac{A}{2} > \cos \frac{A}{2}$ and is positive;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A} = \sqrt{1 + \frac{24}{25}} = \frac{7}{5},$$

and

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 - \frac{24}{25}} = \frac{1}{5};$$

$$\therefore \sin \frac{A}{2} = \frac{4}{5}, \cos \frac{A}{2} = \frac{3}{5}.$$

8. When $\frac{A}{2}$ lies between 135° and 180° , $\cos \frac{A}{2} > \sin \frac{A}{2}$ and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A} = -\sqrt{1 - \frac{240}{289}} = -\frac{7}{17},$$

and

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 + \frac{240}{289}} = \frac{23}{17};$$

$$\therefore \sin \frac{A}{2} = \frac{8}{17}, \cos \frac{A}{2} = -\frac{15}{17}.$$

9. (1) We have

$$\sin A - \cos A = -\sqrt{1 - \sin 2A} \quad \dots \dots \dots \text{(ii)}$$

From (i) we see that of $\sin A$ and $\cos A$ the numerically greater is positive.

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and positive between the limits $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$.

- (3) We have

$$\sin A - \cos A = +\sqrt{1 - \sin 2A} \dots \dots \dots \text{(ii).}$$

From (i) we see that of $\sin A$ and $\cos A$ the numerically greater is negative.

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and negative between the limits $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

10. If $\frac{A}{2} = 120^\circ$, $\sin \frac{A}{2} > \cos \frac{A}{2}$ and is positive,

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A},$$

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A}.$$

$$\therefore 2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}.$$

$$11. \quad \tan 7\frac{1}{2}^\circ = \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \operatorname{cosec} 15^\circ - \cot 15^\circ$$

$$= \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} - (2+\sqrt{3})$$

$$= \sqrt{6} + \sqrt{2} - 2 - \sqrt{3};$$

$$\cot 142\frac{1}{2}^\circ = \frac{1 + \cos 285^\circ}{\sin 285^\circ} = \operatorname{cosec} 285^\circ + \cot 285^\circ$$

$$= -\operatorname{cosec} 75^\circ - \cot 75^\circ = -\frac{2\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} - (2-\sqrt{3})$$

$$= -\sqrt{2}(\sqrt{3}-1) - 2 + \sqrt{3} = \sqrt{2} + \sqrt{3} - 2 - \sqrt{6}.$$

$$12. \quad \sin 9^\circ = \frac{1}{4} \{ \sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}} \} \quad [\text{Art. 260.}]$$

$$= \frac{1}{4} \{ \sqrt{5.236080} - \sqrt{2.7639320} \}$$

$$= \frac{1}{4} \{ 2.288\dots - 1.662\dots \}$$

$$= \frac{.626\dots}{4} = .156\dots$$

$$13. \quad (1) \text{ As in Art. 251, } 2 \cos \frac{\pi}{8} = \sqrt{2+\sqrt{2}};$$

$$\text{but} \quad 4 \sin^2 \frac{\pi}{16} = 2 - 2 \cos \frac{\pi}{8} = 2 - \sqrt{2+\sqrt{2}};$$

$$\therefore 2 \sin \frac{\pi}{16} = \sqrt{2 - \sqrt{2+\sqrt{2}}}.$$

$$(2) \quad \tan 11^\circ 15' = \frac{1 - \cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \operatorname{cosec} 22\frac{1}{2}^\circ - \cot 22\frac{1}{2}^\circ$$

$$= \frac{2}{\sqrt{2}-\sqrt{2}} - \frac{1}{\sqrt{2}-1} = \frac{2}{\sqrt{2}} \frac{\sqrt{2}+\sqrt{2}}{\sqrt{2}} - (\sqrt{2}+1)$$

$$= \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1).$$

$$14. \quad (1) \quad \cos \theta + \sin \theta = \sqrt{2} \cos \left(\theta - \frac{\pi}{4} \right).$$

As θ increases from 0 to $\frac{\pi}{4}$, the expression is positive and increases from 1 to $\sqrt{2}$.

As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the expression is positive and decreases from $\sqrt{2}$ to 0.

As θ increases from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, the expression is negative and increases numerically from 0 to $-\sqrt{2}$.

As θ increases from $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, the expression is negative and decreases numerically from $-\sqrt{2}$ to 0.

As θ increases from $\frac{7\pi}{4}$ to 2π , the expression is positive and increases from 0 to 1.

$$(2) \sin \theta - \sqrt{3} \cos \theta = 2 \left(\frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta \right) = 2 \sin \left(\theta - \frac{\pi}{3} \right).$$

From $\theta=0$ to $\frac{\pi}{3}$, the expression is negative and decreases numerically from $-\sqrt{3}$ to 0.

From $\theta=\frac{\pi}{3}$ to $\frac{\pi}{2} + \frac{\pi}{3}$, or $\frac{5\pi}{6}$, the expression is positive and increases from 0 to 2.

From $\theta=\frac{5\pi}{6}$ to $\frac{\pi}{2} + \frac{5\pi}{6}$, or $\frac{4\pi}{3}$, the expression is positive and decreases from 2 to 0.

From $\theta=\frac{4\pi}{3}$ to $\frac{\pi}{2} + \frac{4\pi}{3}$, or $\frac{11\pi}{6}$, the expression is negative and increases numerically from 0 to -2 .

From $\theta=\frac{11\pi}{6}$ to 2π , the expression is negative and decreases numerically from -2 to $-\sqrt{3}$.

$$15. (1) \text{ The expression } = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} = -\frac{1}{\cos 2\theta} = -\sec 2\theta \\ = -\sec \phi, \text{ where } \phi = 2\theta.$$

We have therefore to trace the changes in $-\sec \phi$, from $\phi=0$ to 2π .

From 0 to $\frac{\pi}{2}$, the expression is negative and increases numerically from -1 to $-\infty$.

From $\frac{\pi}{2}$ to π , the expression is positive and decreases from ∞ to 1.

From π to $\frac{3\pi}{2}$, the expression is positive and increases from 1 to ∞ .

From $\frac{3\pi}{2}$ to 2π , the expression is negative and decreases numerically from $-\infty$ to -1 .

$$(2) \text{ The expression } = \frac{2 \sin \theta (1 - \cos \theta)}{2 \sin \theta (1 + \cos \theta)} = \tan^2 \frac{\theta}{2}.$$

Now $\frac{\theta}{2}$ varies from 0 to $\frac{\pi}{2}$, so that the expression is positive and increases from 0 to ∞ .

EXAMPLES. XX. b. PAGE 260.

1. Here $\tan \frac{A}{2}$ is negative; hence in the formula $\frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}$, the numerator and denominator must have different signs. But when $A = 320^\circ$, $\tan A$ is negative; therefore we must take the sign which will make the numerator positive,

$$\therefore \tan \frac{A}{2} = \frac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$

3. $\tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$, and since $\tan A$ is positive the radical must have the positive sign prefixed.

$$\therefore \tan A = \sqrt{\frac{13 - 12}{13 + 12}} = \frac{1}{5}.$$

4. $\cot \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{1 - \cos A}}$; and when $\frac{A}{2}$ lies between 90° and 135° , $\cot \frac{A}{2}$ is negative;

$$\therefore \cot \frac{A}{2} = - \sqrt{\frac{5 - 4}{5 + 4}} = - \frac{1}{3}.$$

5. The general solution of $\cot 2\theta = \cot 2a$ is

$$2\theta = n\pi + 2a; \text{ therefore } \theta = \frac{1}{2}(n\pi + 2a),$$

$$(1) \text{ when } n = 2m, \quad \cot \theta = \cot(m\pi + a) = \cot a,$$

$$(2) \text{ when } n = 2m + 1, \quad \cot \theta = \cot\left(m\pi + \frac{\pi}{2} + a\right) = -\tan a.$$

6. $\sin \theta = \sin a$; $\therefore \theta = n\pi + (-1)^n a$; hence in finding $\sin \frac{\theta}{3}$ we have to consider all the values of $\sin\left(\frac{n\pi}{3} + (-1)^n \frac{a}{3}\right)$. Give to n in succession the values 0, 1, 2, 3, ... Proceeding as in Art. 264 we shall find that the first six angles are

$$\frac{a}{3}, \quad \frac{\pi - a}{3}, \quad \frac{2\pi + a}{3}, \quad \pi - \frac{a}{3}, \quad \frac{4\pi + a}{3}, \quad \frac{5\pi - a}{3},$$

and that the other angles are coterminal with one of these.

Now

$$\sin \frac{2\pi}{3} + \frac{\alpha}{3} = \sin \left\{ \pi - \left(\frac{\pi}{3} - \frac{\alpha}{3} \right) \right\} = \sin \frac{\pi - \alpha}{3};$$

$$\sin \left(\pi - \frac{\alpha}{3} \right) = \sin \frac{\alpha}{3};$$

$$\sin \left(\frac{4\pi}{3} + \frac{\alpha}{3} \right) = \sin \left\{ \pi + \left(\frac{\pi}{3} + \frac{\alpha}{3} \right) \right\} = -\sin \frac{\pi + \alpha}{3};$$

$$\sin \left(\frac{5\pi}{3} - \frac{\alpha}{3} \right) = \sin \left\{ 2\pi - \left(\frac{\pi}{3} + \frac{\alpha}{3} \right) \right\} = -\sin \frac{\pi + \alpha}{3}.$$

Thus the values of $\sin \frac{\theta}{3}$ are $\sin \frac{\alpha}{3}$, $\sin \frac{\pi - \alpha}{3}$, $-\sin \frac{\pi + \alpha}{3}$.

7. Here $\theta = n\pi + \alpha$, and we have to find all the values of $\tan \left(\frac{n\pi}{3} + \frac{\alpha}{3} \right)$.

Give to n in succession the values 0, 1, 2, 3, ...; then we shall find that all the angles are coterminal with one of the following:

$$\frac{\alpha}{3}, \quad \frac{\pi}{3} + \frac{\alpha}{3}, \quad \frac{2\pi}{3} + \frac{\alpha}{3}, \quad \pi + \frac{\alpha}{3}, \quad \frac{4\pi}{3} + \frac{\alpha}{3}, \quad \frac{5\pi}{3} + \frac{\alpha}{3},$$

and as in Ex. 6 it may be shewn that the tangents of these angles assume one of the three forms

$$\tan \frac{\alpha}{3}, \quad \tan \frac{\pi + \alpha}{3}, \quad -\tan \frac{\pi - \alpha}{3}.$$

8. Here $3\theta = 2n\pi \pm 3\alpha$, or $\theta = \frac{2n\pi}{3} \pm \alpha$, and we have to find all the values of $\sin \left(\frac{2n\pi}{3} \pm \alpha \right)$.

Now n must be of the form $3m$, or $3m+1$, or $3m-1$.

$$\text{If } n=3m, \quad \sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin (2m\pi \pm \alpha) = \pm \sin \alpha;$$

$$\text{if } n=3m+1, \quad \sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin \left(2m\pi + \frac{2\pi}{3} \pm \alpha \right) = \sin \left(\frac{2\pi}{3} \pm \alpha \right);$$

$$\text{if } n=3m-1, \quad \sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin \left(2m\pi - \frac{2\pi}{3} \pm \alpha \right) = -\sin \left(\frac{2\pi}{3} \pm \alpha \right).$$

9. Here $3\theta = n\pi + (-1)^n 3\alpha$, or $\theta = \frac{n\pi}{3} + (-1)^n \alpha$, and we have to find all

the values of $\cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\}$.

$$\text{If } n=3m, \quad \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \{ m\pi + (-1)^{3m} \alpha \} = \pm \cos \alpha;$$

$$\text{if } n=3m+1, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \left\{ m\pi + \frac{\pi}{3} + (-1)^{3m+1} \alpha \right\}$$

$$= \pm \cos \left(\frac{\pi}{3} \pm \alpha \right);$$

$$\text{if } n=3m+2, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \left\{ m\pi + \frac{2\pi}{3} + (-1)^{3m+2} \alpha \right\}$$

$$= \pm \cos \left(\frac{2\pi}{3} \pm \alpha \right).$$

EXAMPLES. XXI. a. PAGE 267.

1. Let x = distance in feet, then $x = 44 \cot 35' = \frac{44}{\theta}$, nearly, where θ is the radian measure of $35'$.

That is, $x = 44 \times \frac{60}{35} \times \frac{180}{\pi} = \frac{44 \times 6 \times 180}{11}$ ft.;

whence $x = 1440$ yds.

2. Here $x = \frac{22}{3} \cot 24' 30'' = \frac{22}{3} \times \frac{180 \times 60}{77}$ ft., nearly
 $= 342 \frac{6}{7}$ yds.

3. With the figure of Ex. 1, p. 264, we have

$$PN = 840 \tan 1^\circ 30' = 840 \times \frac{3}{2} \times \frac{\pi}{180} \text{ yds., nearly}$$

$$= 840 \times \frac{11}{420} = 22 \text{ yds.}$$

4. Let θ be the required angle, then approximately

$$\begin{aligned} \theta = \tan \theta &= \frac{121}{1760 \times 3 \times 12} = \frac{11}{160 \times 3 \times 12} \text{ radians} \\ &= \frac{11}{160 \times 3 \times 12} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} \\ &= 6' 34''. \end{aligned}$$

5. Let θ be the required angle, then approximately

$$\begin{aligned} \frac{\theta}{2} = \tan \frac{\theta}{2} &= \frac{2}{3000} = \frac{1}{1500} \text{ radians;} \\ \therefore \theta &= \frac{1}{750} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} = 4' 35''. \end{aligned}$$

6. The radian measure of $\frac{1^\circ}{4} = \frac{22}{4 \times 180 \times 7} = \frac{11}{14 \times 180}$;

$$\therefore x = .625 \cot \frac{1^\circ}{4} = \frac{14 \times 180}{11} \times .625 \text{ inches, nearly}$$

$$= 11 \text{ ft. } 11 \text{ in.}$$

7. The radian measure of $5' = \frac{5 \times 22}{60 \times 180 \times 7} = \frac{11}{42 \times 180}$;

$$\therefore x = \frac{11}{2} \cot \frac{5'}{2} = \frac{11}{2} \times \frac{84 \times 180}{11} \text{ inches, nearly}$$

$$= 210 \text{ yards.}$$

8. Let θ be the difference between the latitudes, then

$$\begin{aligned}\theta &= \frac{11}{3960} = \frac{1}{360} \text{ radians} \\ &= \frac{180 \times 7 \times 60}{360 \times 22} \text{ minutes} = 9' 33''.\end{aligned}$$

9. See figure of Example 2, page 264.

Let DC be the man, CB the tower, A the point distant 24 feet from the tower;

Let $\angle BAC = \alpha$, $\angle CAD = \theta$;

then $\tan \alpha = \frac{120}{24} = 5$,

$$\tan(\alpha + \theta) = \frac{126}{24} = \frac{21}{4}.$$

But $\tan(\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{\tan \alpha + \theta}{1 - \theta \tan \alpha}$, approximately;

$$\therefore \frac{21}{4} = \frac{5 + \theta}{1 - 5\theta};$$

whence $\theta = \frac{1}{109}$ radians = $31.5'$, nearly.

10. See figure of Example 2, page 264.

Let DC be the flagstaff, CB the cliff, and let $DC = x$ feet; then taking the angles as before, we have

$$\tan \alpha = \frac{490}{980} = \frac{1}{2} = .5, \tan \theta = .04 \text{ approximately};$$

$$\therefore \tan(\alpha + \theta) = \frac{.5 + .04}{1 - .02} = \frac{54}{98} = \frac{27}{49};$$

$$\therefore \frac{x + 490}{980} = \frac{27}{49};$$

whence $x = 50$;

that is, the height of the flagstaff is 50 feet.

$$11. \quad (1) \quad n' = \frac{n\pi}{180 \times 60} \text{ radians;}$$

$$\therefore Lt. \left(\frac{\sin n'}{n} \right) = \frac{n\pi}{180 \times 60} \times \frac{1}{n} = \frac{\pi}{10800}.$$

$$(2) \quad n'' = \frac{n\pi}{180 \times 60 \times 60} \text{ radians;}$$

$$\therefore Lt. \left(\frac{\sin n''}{n} \right) = \frac{n\pi}{180 \times 60 \times 60} \times \frac{1}{n} = \frac{\pi}{648000}.$$

$$12. \quad \frac{1}{2} nr^2 \sin \frac{2\pi}{n} = \pi r^2 \cdot \frac{n}{2\pi} \cdot \sin \frac{2\pi}{n} = \pi r^2 \left(\sin \frac{2\pi}{n} \div \frac{2\pi}{n} \right);$$

but when $n = \infty$, $\frac{2\pi}{n} = 0$, therefore the limit of $\sin \frac{2\pi}{n} \div \frac{2\pi}{n}$ is unity.

Thus the required limit is πr^2 .

$$13. \quad Lt. \left(\frac{1 - \cos \theta}{\theta \sin \theta} \right) = Lt. \left(\frac{\tan \frac{\theta}{2}}{\theta} \right) = Lt. \left(\frac{1}{2} \cdot \frac{\tan \frac{\theta}{2}}{\frac{\theta}{2}} \right) = \frac{1}{2}.$$

$$14. \quad Lt. \left(\frac{m \sin m\theta - n \sin n\theta}{\tan m\theta + \tan n\theta} \right) = Lt. \left(\frac{m \cdot m\theta - n \cdot n\theta}{m\theta + n\theta} \right) \quad [\text{Art. 268.}]$$

$$= \frac{m^2 - n^2}{m - n} = m + n.$$

$$15. \quad \cos \left(\frac{\pi}{3} + \theta \right) = \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2} - \frac{\sqrt{3}}{200} = .491 \text{ nearly.}$$

$$16. \quad 10' 30'' = \frac{21}{2 \times 60 \times 180} \times \frac{22}{7} = \frac{11}{3600} \text{ radians;}$$

$$\therefore \sin 30^\circ 10' 30'' = \sin \left(\frac{\pi}{6} + \frac{11}{3600} \right)$$

$$= \frac{1}{2} \cos \frac{11}{3600} + \frac{\sqrt{3}}{2} \sin \frac{11}{3600}$$

$$= \frac{1}{2} + \frac{11\sqrt{3}}{7200} = .503 \text{ nearly.}$$

17. $\cos\left(\frac{\pi}{3} + \theta\right) = .49$; and $\cos\frac{\pi}{3} = .5$.

$\therefore \theta$ is a very small angle, so that approximately

$$\cos\left(\frac{\pi}{3} + \theta\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\theta;$$

$$\therefore \frac{1}{2} - \frac{\sqrt{3}}{2}\theta = .49;$$

$$\therefore \frac{\sqrt{3}\theta}{2} = .01;$$

$$\therefore \theta = \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{150} \text{ radians}$$

$$= \frac{\sqrt{3}}{150} \times \frac{7}{22} \times 180^\circ = \frac{21\sqrt{3}}{55} \text{ degrees}$$

$$= 39.7' \text{ nearly.}$$

EXAMPLES. XXI. b. PAGE 271.

1. Let the distance be x miles; then by the rule on page 269, we have

$$x^2 = \frac{3 \times 96}{2} = 3 \times 48;$$

$$\therefore x = 12;$$

that is, the distance is 12 miles.

2. Let a feet be the height of lighthouse above the sea level;

then $15^2 = \frac{3a}{2}$, or $a = 150$;

that is, the height of lighthouse = 150 feet.

3. Let the distances in miles of the horizon visible from the masts of the ships be x_1, x_2 ;

then $x_1^2 = \frac{3 \times 32^2}{2} = 49$; $\therefore x_1 = 7$,

$$x_2^2 = \frac{3 \times 42^2}{2} = 64; \therefore x_2 = 8;$$

\therefore the greatest distance at which one mast can be seen from the other

$$= x_1 + x_2 = 15 \text{ miles.}$$

4. Let the distances in miles of the horizon seen from the two masts be x, y respectively,

then $x^2 = \frac{3 \times 54}{2} = 81; \therefore x = 9;$

also $x + y = 20; \text{ whence } y = 11.$

Height of mast of second ship $= \frac{2y^2}{3}$ feet

$$= \frac{242}{3} \text{ feet} = 80 \text{ ft. 8 in.}$$

5. Let x, y be the distances in miles of the horizon visible from the mast and the lamp respectively;

then $x^2 = \frac{3 \times 73\frac{1}{2}}{2}; \therefore x = \frac{21}{2},$

and $x + y = 28; \text{ whence } y = \frac{35}{2}.$

$$\therefore \text{height of lamp} = \frac{2y^2}{3} = \frac{35^2}{6} = \frac{1225}{6} \text{ ft.}$$

$$= 204 \text{ ft. 2 in.}$$

6. From the formula on page 271, we have

$$\text{number of degrees in dip of horizon} = \frac{10}{11} \sqrt{\frac{2 \times 2640}{3 \times 1760}} = \frac{10}{11};$$

$$\therefore \text{dip of the horizon} = 54' 33'', \text{ nearly.}$$

7. The greatest distance at which the light must be visible is the distance of a point exactly opposite a point on the shore midway between two lighthouses, and $3\frac{1}{2}$ miles from it.

$$\text{This distance} = \sqrt{12^2 + \left(\frac{7}{2}\right)^2} = \sqrt{\frac{625}{4}} = \frac{25}{2} \text{ miles.}$$

$$\therefore \text{height of lamp} = \frac{2}{3} \times \frac{625}{4} = \frac{625}{6} \text{ feet} = 104 \text{ ft. 2 in.}$$

8. Let the height of the hill be h miles,

then we have $1.81 = \frac{10}{11} \sqrt{2h};$

$$\therefore \frac{20}{11} = \frac{10}{11} \sqrt{2h}; \text{ whence } h = 2;$$

that is, the height of the hill = 2 miles = 10560 feet.

9. Height of hill = $\frac{2 \times (30 \cdot 25)^2}{3}$ feet = 610 ft. nearly.

The dip of the horizon = $\frac{10}{11} \sqrt{\frac{4 \times 121^2}{3 \times 16} \times \frac{1}{3 \times 1760}}$ degrees
 $= \frac{5}{12} \sqrt{\frac{11}{10}}$ degrees
 $= \frac{5}{2} \sqrt{110}$ minutes
 $= 26' 13''$, nearly.

10. We have $N = \frac{10}{11} \sqrt{2h}$, where h = height in miles,

$$\begin{aligned} &= \frac{10}{11} \sqrt{\frac{2a}{3 \times 1760}}, \text{ where } a = \text{height in feet}, \\ &= \frac{10}{11} \sqrt{\frac{4x^2}{9 \times 1760}} \\ &= \frac{x}{66} \sqrt{\frac{10}{11}}. \end{aligned}$$

11. $\frac{\sin 4\theta \cot \theta}{\operatorname{vers} 2\theta \cot^2 2\theta} = \frac{2 \sin 2\theta \cos 2\theta \cot \theta}{2 \sin^2 \theta \cot^2 2\theta}$

$$= \frac{\sin^3 2\theta \cos \theta}{\sin^3 \theta \cos 2\theta} = \frac{8 \cos^4 \theta}{\cos 2\theta};$$

$$\therefore Lt. \left(\frac{\sin 4\theta \cot \theta}{\operatorname{vers} 2\theta \cot^2 2\theta} \right)_{\theta=0} = Lt. \left(\frac{8 \cos^4 \theta}{\cos 2\theta} \right)_{\theta=0} = 8.$$

12. $\frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} = \frac{2 \sin^2 \frac{\theta}{2} + \sin \theta}{2 \sin^2 \frac{\theta}{2} - \sin \theta} = \frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}};$

$$\therefore Lt. \left(\frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} \right)_{\theta=0} = Lt. \left(\frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}} \right)_{\theta=0} = -1.$$

$$13. \quad (1) \quad Lt._{\theta=\alpha} \left(\frac{\sin \theta - \sin \alpha}{\theta - \alpha} \right) = Lt._{\theta=\alpha} \left(\frac{\frac{\sin \frac{\theta-\alpha}{2} \cos \frac{\theta+\alpha}{2}}{\frac{\theta-\alpha}{2}}}{\frac{\theta-\alpha}{2}} \right).$$

But $Lt._{\theta=\alpha} \left(\frac{\sin \frac{\theta-\alpha}{2}}{\frac{\theta-\alpha}{2}} \right) = 1,$ [Art. 266.]

$$\therefore \text{required limit} = Lt._{\theta=\alpha} \left(\cos \frac{\theta+\alpha}{2} \right) = \cos \alpha.$$

$$(2) \quad Lt._{\theta=\alpha} \left(\frac{\cos \theta - \cos \alpha}{\theta - \alpha} \right) = Lt._{\theta=\alpha} \left(\frac{\frac{\sin \frac{\alpha-\theta}{2} \sin \frac{\alpha+\theta}{2}}{\frac{\theta-\alpha}{2}}}{\frac{\theta-\alpha}{2}} \right)$$

$$= Lt._{\theta=\alpha} \left(-\sin \frac{\alpha+\theta}{2} \right) = -\sin \alpha.$$

14. Let $AB = 32, AC = 31;$

then $\tan C = \frac{32}{31} = 1 + \frac{1}{31};$

$\therefore C$ is a little greater than $45^\circ;$

$$\therefore C = \frac{\pi}{4} + \theta, \text{ where } \theta \text{ is small;}$$

$$\therefore \frac{1+\theta}{1-\theta} = \frac{32}{31};$$

$$\therefore \theta = \frac{1}{63} \text{ radians} = \frac{1}{63} \times \frac{7 \times 180}{22} \text{ degrees} = \frac{10^\circ}{11} = 54' 33'';$$

$$\therefore C = 45^\circ 54' 33'', \quad B = 44^\circ 5' 27''.$$

15. We have

$$\angle BPA = \alpha = \angle BAP;$$

$$\therefore AB = BP;$$

$$\therefore \frac{AB}{BC} = \frac{BP}{BC} = \frac{\sin 3\alpha}{\sin \alpha} = 3 - 4 \sin^2 \alpha = 3,$$

since the object is distant and therefore α is small;

that is,

$$AB = 3BC, \text{ nearly.}$$

16. We shall first shew that $\frac{\tan(\theta+h)}{\theta+h} - \frac{\tan\theta}{\theta}$ is positive, h being the radian measure of a small positive angle.

$$\text{This fraction} = \frac{\theta \tan(\theta+h) - (\theta+h) \tan\theta}{\theta(\theta+h)} = \frac{\theta(\tan\theta+h - \tan\theta) - h \tan\theta}{\theta(\theta+h)}$$

$$= \frac{\frac{\theta \sin h}{\cos\theta \cos(\theta+h)} - \frac{h \sin\theta}{\cos\theta}}{\theta(\theta+h)} = \frac{\theta \sin h - h \sin\theta \cos(\theta+h)}{\theta(\theta+h) \cos\theta \cos(\theta+h)}.$$

$$\text{Now } \frac{\sin h}{h} > \frac{\sin\theta}{\theta} \text{ if } h < \theta$$

[Art. 272.]

and

$$\cos(\theta+h) < 1;$$

$$\therefore \theta \sin h > h \sin\theta \cos(\theta+h);$$

that is, the fraction is positive, and $\therefore \frac{\tan(\theta+h)}{\theta+h} > \frac{\tan\theta}{\theta}$.

$\therefore \frac{\tan\theta}{\theta}$ continually increases as θ increases.

When $\theta=0$, $\frac{\tan\theta}{\theta}=1$; when $\theta=\frac{\pi}{2}$, $\frac{\tan\theta}{\theta}=\infty$.

Thus the proposition is established.

MISCELLANEOUS EXAMPLES. H. PAGE 283.

$$1. \cos 2\alpha + \cos 2\beta + 2 \cos(\alpha + \beta) = 2 \cos(\alpha + \beta) \{ \cos(\alpha - \beta) + 1 \}$$

$$= 4 \cos(\alpha + \beta) \cos^2 \frac{\alpha - \beta}{2},$$

$$\sin 2\alpha + \sin 2\beta + 2 \sin(\alpha + \beta) = 2 \sin(\alpha + \beta) \{ \cos(\alpha - \beta) + 1 \}$$

$$= 4 \sin(\alpha + \beta) \cos^2 \frac{\alpha - \beta}{2};$$

$$\therefore \text{hypotenuse} = 4 \cos^2 \frac{\alpha - \beta}{2} \sqrt{\cos^2(\alpha + \beta) + \sin^2(\alpha + \beta)} = 4 \cos^2 \frac{\alpha - \beta}{2}.$$

2. If the in-centre and circum-centre are at equal distances from BC , we have

$$R \cos A = r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2};$$

$$\therefore \cos A = \cos A + \cos B + \cos C - 1;$$

$$\therefore \cos B + \cos C = 1.$$

3. Let θ be the required angle; then we have

$$\tan(45^\circ + \theta) = 2;$$

$$\therefore \log \tan(45^\circ + \theta) = .3010300$$

$$\begin{array}{r} \log \tan 63^\circ 26' = .3009994 \\ \text{diff.} \qquad \qquad \qquad 306 \\ \hline \end{array}$$

$$\therefore \text{prop'l. increase} = \frac{306}{3159} \times 60'' = 5.8'' = 6'', \text{ nearly};$$

$$\therefore 45^\circ + \theta = 63^\circ 26' 6'';$$

$$\therefore \text{required angle } 18^\circ 26' 6''.$$

4. Denote each expression by E ; then

$$E^2 = (1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \gamma) = \cos^2 \alpha \cos^2 \beta \cos^2 \gamma;$$

that is,

$$E = \pm \cos \alpha \cos \beta \cos \gamma.$$

5. Let p_1, p_2 be the distances of the chords from the centre, and let r be the radius; then

$$p_1 = r \cos 36^\circ;$$

similarly

$$p_2 = r \cos 72^\circ;$$

$$\therefore \text{distance between the chords} = p_1 - p_2 = r(\cos 36^\circ - \cos 72^\circ)$$

$$= r \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{r}{2}.$$

$$\text{Also the sum of the squares of the chords} = 4r^2 \sin^2 36^\circ + 4r^2 \sin^2 72^\circ$$

$$\begin{aligned} &= 4r^2 \left(1 - \frac{3+\sqrt{5}}{8} + 1 - \frac{3-\sqrt{5}}{8} \right) \\ &= 5r^2. \end{aligned}$$

6. Let A be the point at which the railways meet, then we have to solve a triangle in which $A = 60^\circ$, $a = 43$, $b = 48$. It is easy to see that the solution is ambiguous; hence from the third figure of Art. 148 we have

$$CD = 48 \sin 60^\circ = 24\sqrt{3}, \quad AD = 48 \cos 60^\circ = 24.$$

Also

$$B_2 D = \sqrt{43^2 - (24\sqrt{3})^2} = \sqrt{121} = 11.$$

$$\therefore AB_1 = 24 + 11 = 35 \text{ miles},$$

$$AB_2 = 24 - 11 = 13 \text{ miles}.$$

7.

$$a = \cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b};$$

$$\therefore \cos a = \frac{xy}{ab} - \sqrt{\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)};$$

$$\therefore \cos^2 a - \frac{2xy}{ab} \cos a + \frac{x^2 y^2}{a^2 b^2} = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right);$$

$$\therefore 1 - \cos^2 a = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos a + \frac{y^2}{b^2};$$

that is,

$$\sin^2 a = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos a + \frac{y^2}{b^2}.$$

8. We have $p = 4R \cos \frac{A}{2}$, $q = 4R \cos \frac{B}{2}$, $r = 4R \cos \frac{C}{2}$;

$$\therefore \frac{a}{p} = \frac{2R \sin \frac{A}{2}}{4R \cos \frac{A}{2}} = \sin \frac{A}{2};$$

$$\begin{aligned} \therefore \frac{a^2}{p^2} + \frac{b^2}{q^2} + \frac{c^2}{r^2} + \frac{2abc}{pqr} &= \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= 1. \end{aligned}$$

[Ex. 13, p. 120.]

9. Let P be the top of the tower, and let x be its height;

then

$$PA = \frac{x}{\sin \alpha}, \quad PB = \frac{x}{\sin \beta};$$

and

$$PA^2 + OA^2 = PO^2 = PB^2 + OB^2;$$

$$\therefore \frac{x^2}{\sin^2 \alpha} + a^2 = \frac{x^2}{\sin^2 \beta} + b^2;$$

$$\therefore x^2 = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin(\alpha + \beta) \sin(\alpha - \beta)};$$

hence the height of the tower

$$= \frac{\sqrt{a^2 - b^2} \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}.$$

10. This example follows readily from the results proved in Examples 18, 24, 25 of XVIII. a.

$$\begin{aligned} r^2 + r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3 - r)^2 + 2r(r_1 + r_2 + r_3) - 2(r_2 r_3 + r_3 r_1 + r_1 r_2) \\ &= 16R^2 + 2 \{(r_1 r_2 + r r_3) + (r_2 r_3 + r r_1) + (r_3 r_1 + r r_2)\} \\ &\quad - 4(r_1 r_2 + r_2 r_3 + r_3 r_1) \\ &= 16R^2 + 2(ab + bc + ca) - 4s^2 \\ &= 16R^2 - a^2 - b^2 - c^2. \end{aligned}$$

11. (1) Let $\angle BAD = \theta$; then $\angle CAD = A - \theta$;

$$\therefore \frac{\sin(A-\theta)}{\sin(A+B)} = \frac{CD}{AD} = \frac{BD}{AD} = \frac{\sin\theta}{\sin B};$$

$$\therefore \frac{\sin(A-\theta)}{\sin\theta} = \frac{\sin(A+B)}{\sin B};$$

$$\therefore \cot\theta - \cot A = \cot A + \cot B;$$

that is,

$$\cot BAD = 2 \cot A + \cot B.$$

(2) Draw AM perpendicular to BC , then

$$\begin{aligned} 2 \cot ADC &= \frac{2DM}{AM} = \frac{2DC - 2MC}{AM} \\ &= \frac{(BC - MC) - MC}{AM} = \frac{BM}{AM} - \frac{MC}{AM} \\ &= \cot B - \cot C. \end{aligned}$$

12. See fig. of Art. 223.

$$\text{Then } \frac{a}{p} = \frac{BG + GC}{OG} = \frac{BG}{OG} + \frac{GC}{OG} = \tan C + \tan B;$$

$$\therefore \frac{a}{p} + \frac{b}{q} - \frac{c}{r} = 2 \tan C,$$

$$\text{and } \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 2(\tan A + \tan B + \tan C);$$

$$\begin{aligned} \therefore 4 \left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r} \right) &= 8 \tan A \tan B \tan C \\ &= \left(\frac{a}{p} + \frac{b}{q} - \frac{c}{r} \right) \left(\frac{b}{q} + \frac{c}{r} - \frac{a}{p} \right) \left(\frac{c}{r} + \frac{a}{p} - \frac{b}{q} \right). \end{aligned}$$

EXAMPLES. XXIII. a. PAGE 291.

1. Here the common difference is 2α ;

$$\therefore S = \frac{\sin n\alpha}{\sin \alpha} \sin \frac{\alpha + 2n-1\alpha}{2} = \frac{\sin^2 n\alpha}{\sin \alpha}.$$

2. Here the common difference is $-\beta$;

$$\therefore S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \frac{\alpha + \alpha - n-1\beta}{2} = \frac{\sin \frac{n\beta}{2} \cos \left(\alpha - \frac{n-1}{2}\beta \right)}{\sin \frac{\beta}{2}}.$$

3. Here the common difference is $-\frac{\pi}{n}$;

$$\therefore S = \frac{\sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} \sin \frac{a + a - (n-1)\frac{\pi}{n}}{2} = \frac{-\cos \left(a + \frac{\pi}{2n}\right)}{\sin \frac{\pi}{2n}}.$$

4. Here the common difference is $\frac{\pi}{k}$;

$$\therefore S = \frac{\sin \frac{n\pi}{2k}}{\sin \frac{\pi}{2k}} \cos \frac{\frac{\pi}{k} + \frac{n\pi}{k}}{2} = \frac{\sin \frac{n\pi}{2k} \cos \frac{(n+1)\pi}{2k}}{\sin \frac{\pi}{2k}}.$$

5. The common difference is $\frac{2\pi}{19}$ and the number of terms is 9;

$$\therefore S = \frac{\sin \frac{9\pi}{19}}{\sin \frac{\pi}{19}} \cos \frac{9\pi}{19} = \frac{\sin \frac{18\pi}{19}}{2 \sin \frac{\pi}{19}} = \frac{\sin \left(\pi - \frac{\pi}{19}\right)}{2 \sin \frac{\pi}{19}} = \frac{1}{2}.$$

6. Here the common difference is $\frac{2\pi}{21}$ and the number of terms is 10;

$$\therefore S = \frac{\sin \frac{10\pi}{21}}{\sin \frac{\pi}{21}} \cos \frac{11\pi}{21} = \frac{\sin \frac{22\pi}{21}}{2 \sin \frac{\pi}{21}} = -\frac{1}{2}.$$

7. The common difference is $\frac{\pi}{n}$;

$$\therefore S = \frac{\sin \frac{(n-1)\pi}{2n}}{\sin \frac{\pi}{2n}} \sin \frac{\pi + (n-1)\pi}{2n}$$

$$= \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2n}\right)}{\sin \frac{\pi}{2n}} \sin \frac{\pi}{2} = \cot \frac{\pi}{2n}.$$

8. The common difference is $\frac{2\pi}{n}$;

$$\begin{aligned}\therefore S &= \frac{\sin \frac{(2n-1)\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{\pi + \{2(2n-1)-1\}\pi}{2n} \\ &= \frac{\sin \left(2\pi - \frac{\pi}{n}\right)}{\sin \frac{\pi}{n}} \cos \frac{(4n-2)\pi}{2n} = -\cos \frac{\pi}{n}.\end{aligned}$$

9. The common difference is $-\alpha$;

$$\therefore S = \frac{\sin n\alpha}{\sin \frac{\alpha}{2}} \sin \frac{n\alpha + (n-2n-1)\alpha}{2} = \sin n\alpha.$$

10. The series may be written

$$\begin{aligned}\sin \theta + \sin (\pi + 2\theta) + \sin (2\pi + 3\theta) + \sin (3\pi + 4\theta) + \dots \\ \therefore S &= \frac{\sin \frac{n(\pi+\theta)}{2}}{\sin \frac{\pi+\theta}{2}} \sin \frac{\theta + (n-1)\pi + n\theta}{2} \\ &= \frac{\sin \frac{n(\pi+\theta)}{2} \sin \left\{ \frac{(n+1)\theta}{2} + \frac{(n-1)\pi}{2} \right\}}{\sin \frac{\pi+\theta}{2}}.\end{aligned}$$

11. The series may be written

$$\begin{aligned}\cos \alpha + \cos (\pi + \alpha - \beta) + \cos (2\pi + \alpha - 2\beta) + \dots \\ \therefore S &= \frac{\sin \frac{n(\pi-\beta)}{2}}{\sin \frac{\pi-\beta}{2}} \cos \frac{\alpha + (n-1)(\pi-\beta) + \alpha}{2} \\ &= \frac{\sin \frac{n(\pi-\beta)}{2} \cos \left\{ \alpha + \frac{(n-1)(\pi-\beta)}{2} \right\}}{\sin \frac{\pi-\beta}{2}}.\end{aligned}$$

12. The series may be written

$$\begin{aligned} & \cos \alpha + \cos \left(\frac{\pi}{2} + \alpha - \beta \right) + \cos (\pi + \alpha - 2\beta) + \cos \left(\frac{3\pi}{2} + \alpha - 3\beta \right) + \dots \\ \therefore S &= \frac{\sin \frac{n(\pi - 2\beta)}{4}}{\sin \frac{\pi - 2\beta}{4}} \cos \frac{2\alpha + (n-1)\left(\frac{\pi}{2} - \beta\right)}{2} \\ &= \frac{\sin \frac{n(\pi - 2\beta)}{4} \cos \left\{ \alpha + \frac{(n-1)(\pi - 2\beta)}{4} \right\}}{\sin \frac{\pi - 2\beta}{4}}. \end{aligned}$$

$$\begin{aligned} 13. \quad S &= \frac{1}{2} \{ (\cos \theta - \cos 3\theta) + (\cos \theta - \cos 5\theta) + (\cos \theta - \cos 7\theta) + \dots \} \\ &= \frac{1}{2} \{ n \cos \theta - (\cos 3\theta + \cos 5\theta + \cos 7\theta + \dots + \cos 2n+1\theta) \} \\ &= \frac{n \cos \theta}{2} - \frac{\sin n\theta}{2 \sin \theta} \cos (n+2)\theta. \end{aligned}$$

$$\begin{aligned} 14. \quad S &= \frac{1}{2} \{ (\sin 4\alpha - \sin 2\alpha) + (\sin 8\alpha - \sin 2\alpha) + (\sin 12\alpha - \sin 2\alpha) + \dots \} \\ &= \frac{1}{2} \{ (\sin 4\alpha + \sin 8\alpha + \sin 12\alpha + \dots + \sin 4n\alpha) - n \sin 2\alpha \} \\ &= \frac{\sin 2n\alpha}{2 \sin 2\alpha} \sin 2(n+1)\alpha - \frac{n \sin 2\alpha}{2}. \end{aligned}$$

$$\begin{aligned} 15. \quad \sec \alpha \sec 2\alpha &= \frac{1}{\cos \alpha \cos 2\alpha} = \operatorname{cosec} \alpha \cdot \frac{\sin (2\alpha - \alpha)}{\cos \alpha \cos 2\alpha} \\ &= \operatorname{cosec} \alpha (\tan 2\alpha - \tan \alpha). \end{aligned}$$

Similarly,

$$\sec 2\alpha \sec 3\alpha = \operatorname{cosec} \alpha (\tan 3\alpha - \tan 2\alpha),$$

.....

$$\sec n\alpha \sec (n+1)\alpha = \operatorname{cosec} \alpha \{ \tan (n+1)\alpha - \tan n\alpha \}.$$

By addition,

$$S = \operatorname{cosec} \alpha \{ \tan (n+1)\alpha - \tan n\alpha \}.$$

16. $\text{cosec } \theta \text{ cosec } 3\theta = \text{cosec } 2\theta (\cot \theta - \cot 3\theta),$
 $\text{cosec } 3\theta \text{ cosec } 5\theta = \text{cosec } 2\theta (\cot 3\theta - \cot 5\theta),$
.....
 $\text{cosec } (2n-1)\theta \text{ cosec } (2n+1)\theta = \text{cosec } 2\theta \{ \cot (2n-1)\theta - \cot (2n+1)\theta \}.$

By addition, $S = \text{cosec } 2\theta \{ \cot \theta - \cot (2n+1)\theta \}.$

17. $\tan \frac{\alpha}{2} \sec \alpha = \tan \alpha - \tan \frac{\alpha}{2},$

$$\tan \frac{\alpha}{2^2} \sec \frac{\alpha}{2} = \tan \frac{\alpha}{2} - \tan \frac{\alpha}{2^2},$$

$$\tan \frac{\alpha}{2^n} \sec \frac{\alpha}{2^{n-1}} = \tan \frac{\alpha}{2^{n-1}} - \tan \frac{\alpha}{2^n}.$$

By addition, $S = \tan \alpha - \tan \frac{\alpha}{2^n}.$

18. $\cos 2\alpha \text{ cosec } 3\alpha = \frac{\cos 2\alpha \sin \alpha}{\sin 3\alpha \sin \alpha} = \frac{1}{2} \cdot \frac{\sin 3\alpha - \sin \alpha}{\sin 3\alpha \sin \alpha}$
 $= \frac{1}{2} (\text{cosec } \alpha - \text{cosec } 3\alpha),$
 $\cos 6\alpha \text{ cosec } 9\alpha = \frac{1}{2} (\text{cosec } 3\alpha - \text{cosec } 9\alpha),$
.....

$$\cos 3^{n-1} \cdot 2\alpha \text{ cosec } 3^n \alpha = \frac{1}{2} (\text{cosec } 3^{n-1} \alpha - \text{cosec } 3^n \alpha).$$

By addition, $S = \frac{1}{2} \text{cosec } \alpha - \text{cosec } 3^n \alpha.$

19. $\sin \alpha \sec 3\alpha = \frac{\sin \alpha}{\cos 3\alpha} = \frac{2 \sin \alpha \cos \alpha}{2 \cos 3\alpha \cos \alpha}$
 $= \frac{\sin (3\alpha - \alpha)}{2 \cos 3\alpha \cos \alpha}$
 $= \frac{1}{2} (\tan 3\alpha - \tan \alpha),$

$$\sin 3\alpha \sec 9\alpha = \frac{1}{2} (\tan 9\alpha - \tan 3\alpha),$$

$$\sin 3^{n-1}\alpha \sec 3^n \alpha = \frac{1}{2} (\tan 3^n \alpha - \tan 3^{n-1} \alpha).$$

By addition, $S = \frac{1}{2} (\tan 3^n \alpha - \tan \alpha).$

20. Let AB be the diameter, $P_1, P_2, P_3 \dots P_{n-1}$ the points of division of the arc of the semicircle starting from the end B ; then

$$AP_1 = 2a \cos \frac{\pi}{2n}, \quad AP_2 = 2a \cos \frac{2\pi}{2n}, \dots \quad AP_{n-1} = 2a \cos \frac{(n-1)\pi}{2n};$$

$$\begin{aligned}\therefore \text{sum of distances} &= 2a \left\{ \cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + \cos \frac{3\pi}{2n} + \dots + \cos \frac{(n-1)\pi}{2n} \right\} \\ &= \frac{2a \sin \frac{(n-1)\pi}{4n}}{\sin \frac{\pi}{4n}} \cos \frac{\pi}{4} \\ &= \frac{a \left\{ \sin \left(\frac{\pi}{2} - \frac{\pi}{4n} \right) - \sin \frac{\pi}{4n} \right\}}{\sin \frac{\pi}{4n}} \\ &= a \left(\cot \frac{\pi}{4n} - 1 \right).\end{aligned}$$

21. Let $P_1, P_2, P_3 \dots$ be the angular points of the polygon beginning with that nearest to XX' in the quadrant XOY .

Let $p_1, p_2, p_3 \dots$ be perpendiculars from $P_1, P_2, P_3 \dots$ on XX' , and $q_1, q_2, q_3 \dots$ be perpendiculars from $P_1, P_2, P_3 \dots$ on YY' .

Let $\angle P_1OX = \theta$, and let r be the radius of the circle;

$$\begin{aligned}\text{then } p_1 &= r \sin \theta, \quad p_2 = r \sin \left(\theta + \frac{2\pi}{n} \right), \quad p_3 = r \sin \left(\theta + \frac{4\pi}{n} \right), \dots; \\ \therefore S_p &= r \left\{ \sin \theta + \sin \left(\theta + \frac{2\pi}{n} \right) + \sin \left(\theta + \frac{4\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ &= r \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cdot \sin \frac{2\theta + (n-1)\frac{2\pi}{n}}{2} = 0.\end{aligned}$$

Similarly $S_q = 0$.

EXAMPLES. XXIII. b. PAGE 294.

$$1. \quad 2S = 1 + \cos 2\theta + 1 + \cos 6\theta + 1 + \cos 10\theta + \dots$$

$$= n + \frac{\sin 2n\theta}{\sin 2\theta} \cos \frac{2\theta + 2\theta + (n-1)4\theta}{2};$$

$$\therefore S = \frac{n}{2} + \frac{\sin 4n\theta}{4 \sin 2\theta}.$$

$$2. \quad 2S = 1 - \cos 2\alpha + 1 - \cos 2\left(\alpha + \frac{\pi}{n}\right) + 1 - \cos 2\left(\alpha + \frac{2\pi}{n}\right) + \dots$$

$$= n - \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{4\alpha + (n-1)\frac{2\pi}{n}}{2} = n.$$

$$3. \quad 2S = 1 + \cos 2\alpha + 1 + \cos 2\left(\alpha - \frac{\pi}{n}\right) + 1 + \cos 2\left(\alpha - \frac{2\pi}{n}\right) + \dots$$

$$= n + \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{4\alpha - (n-1)\frac{2\pi}{n}}{2} = n.$$

$$4. \quad 4S = 3 \sin \theta - \sin 3\theta + 3 \sin 2\theta - \sin 6\theta + 3 \sin 3\theta - \sin 9\theta + \dots$$

$$= 3 \{ \sin \theta + \sin 2\theta + \sin 3\theta + \dots \} - (\sin 3\theta + \sin 6\theta + \sin 9\theta + \dots)$$

$$= \frac{3 \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \frac{(n+1)\theta}{2} - \frac{\sin \frac{3n\theta}{2}}{\sin \frac{3\theta}{2}} \sin \frac{3(n+1)\theta}{2};$$

$$\therefore S = \frac{3 \sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{4 \sin \frac{\theta}{2}} - \frac{\sin \frac{3n\theta}{2} \sin \frac{3(n+1)\theta}{2}}{\sin \frac{3\theta}{2}}.$$

$$5. \quad 4S = 3 \left\{ \sin \alpha + \sin \left(\alpha + \frac{2\pi}{n} \right) + \sin \left(\alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$- \left\{ \sin 3\alpha + \sin 3\left(\alpha + \frac{2\pi}{n} \right) + \sin 3\left(\alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$= 3 \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \sin \frac{2\alpha + (n-1)\frac{2\pi}{n}}{2} - \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \sin \frac{6\alpha + (n-1)\frac{6\pi}{n}}{2}$$

$$= 0.$$

$$\begin{aligned}
 6. \quad 4S &= 3 \left\{ \cos \alpha + \cos \left(\alpha - \frac{2\pi}{n} \right) + \cos \left(\alpha - \frac{4\pi}{n} \right) + \dots \right\} \\
 &\quad + \cos 3\alpha + \cos 3 \left(\alpha - \frac{2\pi}{n} \right) + \cos 3 \left(\alpha - \frac{4\pi}{n} \right) + \dots \\
 &= 3 \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{2\alpha - (n-1)\frac{2\pi}{n}}{2} + \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \cos \frac{6\alpha - (n-1)\frac{6\pi}{n}}{2} \\
 &= 0.
 \end{aligned}$$

7. We have

$$\tan \theta = \cot \theta - 2 \cot 2\theta,$$

[XI. d. Ex. 18.]

$$2 \tan 2\theta = 2 \cot 2\theta - 2^2 \cot 2^2 \theta,$$

$$2^2 \tan 2^2 \theta = 2^2 \cot 2^2 \theta - 2^3 \cot 2^3 \theta,$$

$$2^{n-1} \tan 2^{n-1} \theta = 2^{n-1} \cot 2^{n-1} \theta - 2^n \cot 2^n \theta;$$

∴ by addition,

$$S = \cot \theta - 2^n \cot 2^n \theta.$$

$$8. \quad S = \frac{1}{2 \cos \alpha \cos 2\alpha} + \frac{1}{2 \cos 2\alpha \cos 3\alpha} + \frac{1}{2 \cos 3\alpha \cos 4\alpha} + \dots$$

$$= \frac{1}{2} \{ \sec \alpha \sec 2\alpha + \sec 2\alpha \sec 3\alpha + \sec 3\alpha \sec 4\alpha + \dots \}$$

$$= \frac{1}{2} \operatorname{cosec} \alpha \{ \tan(n+1)\alpha - \tan \alpha \}. \quad [\text{XXX}]$$

0

$$\sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} (1 - \cos 2\theta);$$

$$\therefore \sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4}.$$

Replacing θ by 2θ , we obtain

$$\frac{1}{2} \sin^2 2\theta \sin 4\theta = \frac{\sin 4\theta}{4} - \frac{\sin 8\theta}{8}.$$

Similarly,

$$\frac{1}{4} \sin^2 4\theta \sin 8\theta = \frac{\sin 8\theta}{8} - \frac{\sin 16\theta}{16};$$

$$\frac{1}{2^{n-1}} \sin^2 2^{n-1} \theta \sin 2^n \theta = \frac{\sin 2^n \theta}{2^n} - \frac{\sin 2^{n+1} \theta}{2^{n+1}};$$

· by addition

$$S = \frac{\sin 2\theta}{2} - \frac{\sin 2n+1 \theta}{2n+1}$$

10. $2 \cos \theta \sin^2 \frac{\theta}{2} = \cos \theta (1 - \cos \theta) = 1 - \cos^2 \theta - (1 - \cos \theta);$

$$\therefore 2 \cos \theta \sin^2 \frac{\theta}{2} = \sin^2 \theta - 2 \sin^2 \frac{\theta}{2}.$$

Replacing θ by $\frac{\theta}{2}$, we obtain

$$2^2 \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2^2} = 2 \sin^2 \frac{\theta}{2} - 2^2 \sin^2 \frac{\theta}{2^2}.$$

Similarly, $2^3 \cos \frac{\theta}{2^2} \sin^2 \frac{\theta}{2^3} = 2^2 \sin^2 \frac{\theta}{2^2} - 2^3 \sin^2 \frac{\theta}{2^3};$

$$2^n \cos \frac{\theta}{2^{n-1}} \sin^2 \frac{\theta}{2^n} = 2^{n-1} \sin^2 \frac{\theta}{2^{n-1}} - 2^n \sin^2 \frac{\theta}{2^n};$$

\therefore by addition, $S = \sin^2 \theta - 2^n \sin^2 \frac{\theta}{2^n}.$

11. We have

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{\frac{x}{n} - \frac{x}{n+1}}{1 + \frac{x^2}{n(n+1)}} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1},$$

and hence

$$\tan^{-1} \frac{x}{1+2+x^2} = \tan^{-1} x - \tan^{-1} \frac{x}{2};$$

$$\tan^{-1} \frac{x}{2+3+x^2} = \tan^{-1} \frac{x}{2} - \tan^{-1} \frac{x}{3};$$

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1};$$

\therefore by addition,

$$S = \tan^{-1} x - \tan^{-1} \frac{x}{n+1}.$$

12. $\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} \frac{(n+1)-n}{1+n(n+1)}$
 $= \tan^{-1} (n+1) - \tan^{-1} n.$

$$\therefore \tan^{-1} \frac{1}{1+1+1^2} = \tan^{-1} 2 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{1}{1+2+2^2} = \tan^{-1} 3 - \tan^{-1} 2;$$

$$\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} (n+1) - \tan^{-1} n;$$

$$\therefore S = \tan^{-1} (n+1) - \tan^{-1} 1 = \tan^{-1} (n+1) - \frac{\pi}{4}.$$

$$13. \quad \tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1} \frac{(1+n+n^2)-(1-n+n^2)}{1+(1+n+n^2)(1-n+n^2)} \\ = \tan^{-1}(1+n+n^2) - \tan^{-1}(1-n+n^2).$$

$$\therefore \tan^{-1} \frac{2}{2+1^2+1^4} = \tan^{-1} 3 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{4}{2+2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 3;$$

.....

$$\tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1}(1+n+n^2) - \tan^{-1}(1-n+n^2); \\ \therefore S = \tan^{-1}(1+n+n^2) - \tan^{-1} 1 \\ = \tan^{-1}(1+n+n^2) - \frac{\pi}{4}.$$

$$14. \quad \tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1} \frac{n^2+n-(n^2-n)}{1+(n^2+n)(n^2-n)} \\ = \tan^{-1}(n^2+n) - \tan^2(n^2-n).$$

$$\therefore \tan^{-1} \frac{2}{1-1^2+1^4} = \tan^{-1} 2 - \tan^{-1} 0;$$

$$\tan^{-1} \frac{4}{1-2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 2;$$

.....

$$\tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1}(n^2+n) - \tan^2(n^2-n); \\ \therefore S = \tan^{-1}(n^2+n).$$

15. Let O be the point on the circumference of the circle, and P, Q, R ... the vertices of the polygon beginning with that nearest to O . Let L be the other extremity of the diameter through O , and let $\angle OLP = \theta$;

then $OP = 2r \sin \theta, OQ = 2r \sin \left(\theta + \frac{\pi}{n} \right), OR = 2r \sin \left(\theta + \frac{2\pi}{n} \right), \dots$

\therefore sum of the squares of the chords

$$= 4r^2 \left\{ \sin^2 \theta + \sin^2 \left(\theta + \frac{\pi}{n} \right) + \sin^2 \left(\theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r^2 \left\{ n - \cos 2\theta - \cos 2 \left(\theta + \frac{\pi}{n} \right) - \cos 2 \left(\theta + \frac{2\pi}{n} \right) - \dots \right\}$$

$$= 2nr^2 - 2r^2 \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \left(2\theta + (n-1) \frac{\pi}{n} \right)$$

$$= 2nr^2.$$

16. Let O be the centre of the inscribed circle, and let $\angle PON_1 = \theta$, where ON_1 is the perpendicular from O parallel to PA_1 ; let $OP = x$;

$$\text{then } PA_1 = r - x \cos \theta, \quad PA_2 = r - x \cos \left(\theta + \frac{\pi}{n} \right),$$

$$PA_3 = r - x \cos \left(\theta + \frac{2\pi}{n} \right), \quad PA_4 = r - x \cos \left(\theta + \frac{3\pi}{n} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n-1}$$

$$= nr - x \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \cos \left(\theta + \frac{4\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= nr.$$

[Art. 297.]

$$\text{Similarly } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= nr - x \left\{ \cos \left(\theta + \frac{\pi}{n} \right) + \cos \left(\theta + \frac{3\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= nr.$$

17. Let Q be the other extremity of the diameter through P , and let $\angle PQA_1 = \theta$, and let r be the radius of the circle; then

$$PA_1 = 2r \sin \theta, \quad PA_2 = 2r \sin \left(\theta + \frac{\pi}{2n+1} \right), \quad PA_3 = 2r \sin \left(\theta + \frac{2\pi}{2n+1} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1}$$

$$= 2r \left\{ \sin \theta + \sin \left(\theta + \frac{2\pi}{2n+1} \right) + \sin \left(\theta + \frac{4\pi}{2n+1} \right) + \dots \text{ to } n+1 \text{ terms} \right\}$$

$$= 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left(\theta + \frac{n\pi}{2n+1} \right);$$

$$\text{and } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= 2r \left\{ \sin \left(\theta + \frac{\pi}{2n+1} \right) + \sin \left(\theta + \frac{3\pi}{2n+1} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r \frac{\sin \frac{n\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \frac{1}{2} \left(2\theta + \frac{\pi}{2n+1} + \frac{(2n-1)\pi}{2n+1} \right)$$

$$= 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left(\theta + \frac{n\pi}{2n+1} \right).$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1} = PA_2 + PA_4 + \dots + PA_{2n}.$$

18. Let $p_1, p_2, p_3, \dots, p_n$ be the perpendiculars; then as in Ex. 16, we have

$$p_1 = r - r \cos \theta, \quad p_2 = r - r \cos \left(\theta + \frac{2\pi}{n} \right), \quad p_3 = r - r \cos \left(\theta + \frac{4\pi}{n} \right), \dots$$

$$\begin{aligned} (i) \quad \Sigma p^2 &= \{r - r \cos \theta\}^2 + \left\{r - r \cos \left(\theta + \frac{2\pi}{n} \right)\right\}^2 + \dots \text{ to } n \text{ terms} \\ &= nr^2 - 2r^2 \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad + r^2 \left\{ \cos^2 \theta + \cos^2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\}. \end{aligned}$$

Now

$$\cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots = 0; \quad [\text{Art. 297.}]$$

$$\begin{aligned} \therefore \Sigma p^2 &= nr^2 + \frac{r^2}{2} \left\{ n + \cos 2\theta + \cos 2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &= nr^2 + \frac{nr^2}{2} = \frac{3nr^2}{2}. \end{aligned}$$

$$\begin{aligned} (ii) \quad \Sigma p^3 &= nr^3 - 3r^3 \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ &\quad + \frac{3r^3}{2} \left\{ n + \cos 2\theta + \cos 2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad - \frac{r^3}{4} \left\{ 3 \cos \theta + 3 \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \right. \\ &\quad \quad \left. + \cos 3\theta + \cos 3 \left(\theta + \frac{3\pi}{n} \right) + \dots \right\} \\ &= nr^3 + \frac{3nr^3}{2} = \frac{5nr^3}{2}, \end{aligned}$$

since all the series of cosines vanish by Art. 297.

EXAMPLES. XXIV. a. PAGE 301.

1. We have

$$\frac{1}{a} \cos \alpha + \frac{1}{b} \sin \alpha = \frac{1}{c}, \quad \frac{1}{a} \cos \beta + \frac{1}{b} \sin \beta = \frac{1}{c};$$

and the required result follows at once by cross multiplication as in Ex. 1, p. 297.

4.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2},$$

as in Example 2, page 297.

$$\begin{aligned}
 5. \quad \cos^2 \frac{\alpha - \beta}{2} &= \frac{1 + \cos(\alpha - \beta)}{2} = \frac{1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta}{2} \\
 &= \frac{(a^2 + b^2) + (c^2 - b^2) + (c^2 - a^2)}{2(a^2 + b^2)} \\
 &= \frac{c^2}{a^2 + b^2}.
 \end{aligned}$$

$$6. \quad \sin 2\alpha + \sin 2\beta = 2 \sin(\alpha + \beta) \cos(\alpha - \beta).$$

From Example 4, we find that $\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$;

$$\text{and } 7. \quad \cos(\alpha - \beta) = \frac{c^2 - b^2}{a^2 + b^2} + \frac{c^2 - a^2}{a^2 + b^2} = \frac{2c^2 - a^2 - b^2}{a^2 + b^2}.$$

$$7. \quad \sin^2 \alpha + \sin^2 \beta = (\sin \alpha + \sin \beta)^2 - 2 \sin \alpha \sin \beta$$

$$= \left(\frac{2bc}{a^2 + b^2} \right)^2 - \frac{2(c^2 - a^2)}{a^2 + b^2}. \quad [\text{See p. 298.}]$$

8. Here α and β are solutions of $a \cos \theta + b \sin \theta = c$; hence as in Example 4, we find

$$\cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}, \text{ and therefore } \sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}.$$

$$\begin{aligned}
 \text{Again, } \cot \alpha + \cot \beta &= \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta} \\
 &= \frac{2ab}{a^2 + b^2} \div \frac{c^2 - a^2}{a^2 + b^2}. \quad [\text{See p. 298.}]
 \end{aligned}$$

9. By squaring and adding, we have

$$2 + 2 \cos(\theta - \phi) = a^2 + b^2.$$

$$\text{Again, } 10. \quad \cos(\theta + \phi) = \frac{a^2 - b^2}{a^2 + b^2}. \quad [\text{Ex. 4, p. 299.}]$$

$$\text{And } 11. \quad 2 \cos \theta \cos \phi = \cos(\theta + \phi) + \cos(\theta - \phi);$$

$$\begin{aligned}
 \therefore 4 \cos \theta \cos \phi &= \frac{2(a^2 - b^2)}{a^2 + b^2} + a^2 + b^2 - 2 \\
 &= \frac{(a^2 + b^2)^2 - 4b^2}{a^2 + b^2}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \cos 2\theta + \cos 2\phi &= 2 \cos(\theta + \phi) \cos(\theta - \phi) \\
 &= \frac{a^2 - b^2}{a^2 + b^2} (a^2 + b^2 - 2),
 \end{aligned}$$

as in the preceding example.

$$11. \quad \tan \theta + \tan \phi = \frac{\sin(\theta + \phi)}{\cos \theta \cos \phi}$$

$$= \frac{2ab}{a^2 + b^2} \div \frac{(a^2 + b^2)^2 - 4b^2}{4(a^2 + b^2)}.$$

[See Ex. 4, p. 299 and Ex. 9 above.]

$$12. \quad \tan \frac{\theta}{2} + \tan \frac{\phi}{2} = \frac{\sin \frac{\theta + \phi}{2}}{\cos \frac{\theta}{2} \cos \frac{\phi}{2}} = \frac{2 \sin \frac{\theta + \phi}{2}}{\cos \frac{\theta + \phi}{2} + \cos \frac{\theta - \phi}{2}}.$$

On multiplying numerator and denominator by $2 \cos \frac{\theta + \phi}{2}$, this last fraction becomes

$$\frac{2 \sin(\theta + \phi)}{1 + \cos(\theta + \phi) + \cos \theta + \cos \phi}.$$

By substituting for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$ the values found in Ex. 4, p. 299, we obtain

$$\frac{4ab}{a^2 + b^2} \div \left(1 + \frac{a^2 - b^2}{a^2 + b^2} + a \right),$$

which reduces to $\frac{4b}{a^2 + b^2 + 2a}$.

$$13. \quad \begin{aligned} & \text{The expression} = \sin^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma \\ & = -\cos(\alpha + \beta) \cos(\alpha - \beta) - \cos^2 \gamma + \{\cos(\alpha + \beta) + \cos(\alpha - \beta)\} \cos \gamma \\ & = -\{\cos(\alpha + \beta) - \cos \gamma\} \{\cos(\alpha - \beta) - \cos \gamma\} \\ & = 4 \sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\beta + \gamma - \alpha}{2} \sin \frac{\gamma + \alpha - \beta}{2}; \end{aligned}$$

from which the second part of the question easily follows.

$$14. \quad \begin{aligned} & \text{The expression} = \sin^2 \alpha + (\sin^2 \beta - \sin^2 \gamma) + 2 \sin \alpha \sin \beta \cos \gamma \\ & = \sin^2 \alpha + \sin(\beta + \gamma) \sin(\beta - \gamma) + \sin \alpha \{\sin(\beta + \gamma) + \sin(\beta - \gamma)\} \\ & = \{\sin \alpha + \sin(\beta + \gamma)\} \{\sin \alpha + \sin(\beta - \gamma)\} \\ & = 4 \sin \frac{\alpha + \beta + \gamma}{2} \cos \frac{\beta + \gamma - \alpha}{2} \sin \frac{\alpha + \beta - \gamma}{2} \cos \frac{\alpha - \beta + \gamma}{2}. \end{aligned}$$

$$\begin{aligned}
 15. \quad & \text{The expression} = \sin^2 \alpha - (\cos^2 \beta - \sin^2 \gamma) - 2 \sin \alpha \sin \beta \sin \gamma \\
 & = \sin^2 \alpha - \cos(\beta + \gamma) \cos(\beta - \gamma) - \sin \alpha \{\cos(\beta - \gamma) - \cos(\beta + \gamma)\} \\
 & = \{\sin \alpha + \cos(\beta + \gamma)\} \{\sin \alpha - \cos(\beta - \gamma)\}; \\
 & = - \left\{ \cos\left(\frac{\pi}{2} - \alpha\right) + \cos(\beta + \gamma) \right\} \left\{ \cos\left(\frac{\pi}{2} + \alpha\right) + \cos(\beta - \gamma) \right\} \\
 & = - 4 \cos\left(\frac{\beta + \gamma - \alpha}{2} + \frac{\pi}{4}\right) \cos\left(\frac{\beta + \gamma + \alpha}{2} - \frac{\pi}{4}\right) \cos\left(\frac{\alpha + \beta - \gamma}{2} + \frac{\pi}{4}\right) \\
 & \quad \cos\left(\frac{\alpha - \beta + \gamma}{2} + \frac{\pi}{4}\right)
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \tan^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - \cos \alpha + \cos \beta - \cos \alpha \cos \beta}{1 + \cos \alpha - \cos \beta - \cos \alpha \cos \beta} \\
 &= \frac{(1 - \cos \alpha)(1 + \cos \beta)}{(1 + \cos \alpha)(1 - \cos \beta)} = \tan^2 \frac{\alpha}{2} \cot^2 \frac{\beta}{2}.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \tan^2 \theta &= \frac{2 \sin \alpha \sin \beta}{1 + \cos(\alpha + \beta)} = \frac{2 \sin \alpha \sin \beta}{1 + \cos \alpha \cos \beta - \sin \alpha \sin \beta}; \\
 \therefore 1 + \tan^2 \theta &= \frac{1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta}{1 + \cos \alpha \cos \beta - \sin \alpha \sin \beta};
 \end{aligned}$$

that is, $\sec^2 \theta = \frac{1 + \cos(\alpha - \beta)}{1 + \cos(\alpha + \beta)} = \frac{\cos^2 \frac{\alpha - \beta}{2}}{\cos^2 \frac{\alpha + \beta}{2}}.$

Taking the positive value of the square root, we have

$$\cos \theta = \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}.$$

$$\therefore \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\alpha}{2} \cos \frac{\beta}{2}}.$$

18. Here

$$\tan \theta = \frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta};$$

$$\begin{aligned}\therefore \sec^2 \theta &= 1 + \frac{\sin^2 \alpha \cos^2 \beta}{(\cos \alpha + \sin \beta)^2} = \frac{(\cos \alpha + \sin \beta)^2 + (1 - \cos^2 \alpha)(1 - \sin^2 \beta)}{(\cos \alpha + \sin \beta)^2} \\ &= \frac{1 + 2 \cos \alpha \sin \beta + \cos^2 \alpha \sin^2 \beta}{(\cos \alpha + \sin \beta)^2}.\end{aligned}$$

Taking the positive value of the square root, we have

$$\cos \theta = \frac{\cos \alpha + \sin \beta}{1 + \cos \alpha \sin \beta}.$$

$$\begin{aligned}\therefore \frac{1 - \cos \theta}{1 + \cos \theta} &= \frac{(1 - \cos \alpha)(1 - \sin \beta)}{(1 + \cos \alpha)(1 + \sin \beta)} \\ &= \frac{(1 - \cos \alpha) \left\{ 1 - \cos \left(\frac{\pi}{2} - \beta \right) \right\}}{(1 + \cos \alpha) \left\{ 1 + \cos \left(\frac{\pi}{2} - \beta \right) \right\}}; \\ \therefore \tan^2 \frac{\theta}{2} &= \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right).\end{aligned}$$

19. This identity will be established if we shew that

$$\Sigma \sin^3 A \sin B \sin C = 2 \sin A \sin B \sin C (1 + \cos A \cos B \cos C),$$

that is, if

$$\Sigma \sin^2 A = 2 (1 + \cos A \cos B \cos C).$$

$$\begin{aligned}\text{Now } \sin^2 A + \sin^2 B + \sin^2 C &= \frac{1}{2} (3 - \cos 2A - \cos 2B - \cos 2C) \\ &= \frac{1}{2} (4 + 4 \cos A \cos B \cos C).\end{aligned}$$

[See XII. d. Ex. 9.]

20. Expressing a, b, c in terms of R , this identity will be proved if we shew that

$$\Sigma \sin A \cos^3 A = \Pi \sin A (1 - 4 \cos A \cos B \cos C).$$

$$\text{Now } 8 \Sigma \sin A \cos^3 A = 4 \Sigma \cos^2 A \sin 2A$$

$$= 2 \Sigma (1 + \cos 2A) \sin 2A$$

$$= 2 \Sigma \sin 2A + \Sigma \sin 4A.$$

$$\text{Now }$$

$$\Sigma \sin 2A = 4 \Pi \sin A,$$

and

$$\Sigma \sin 4A = -4 \Pi \sin 2A;$$

[Ex. 7, p. 301];

$$\therefore \Sigma \sin A \cos^3 A = \Pi \sin A - 4 \Pi \sin A \cos A$$

$$= \Pi \sin A (1 - 4 \cos A \cos B \cos C).$$

$$\begin{aligned} 21. \quad \Sigma a^3 \cos(B-C) &= 2R\Sigma a^2 \sin A \cos(B-C) \\ &= 2R\Sigma a^2 \sin(B+C) \cos(B-C) \\ &= R\Sigma a^2 (\sin 2B + \sin 2C). \end{aligned}$$

$$\begin{aligned} \text{Now } a^2 \sin 2B + b^2 \sin 2A &= 2a \sin B \cdot a \cos B + 2b \sin A \cdot b \cos A \\ &= 2a \sin B (a \cos B + b \cos A) \\ &= 2ac \sin B = 4\Delta. \end{aligned}$$

$$\text{Hence } \Sigma a^3 \cos(B - C) = 12R\Delta = 3abc.$$

22. (1) We have, from the given equation,

$$(b \sin \theta - c)^2 = a^2 \cos^2 \theta = a^2 (1 - \sin^2 \theta),$$

$$\text{or} \quad (a^2 + b^2) \sin^2 \theta - 2bc \sin \theta + (c^2 - a^2) = 0;$$

and this is the required equation since by hypothesis it is satisfied by $\sin \alpha$ and $\sin \beta$.

(2) The required equation is

$$x^2 - (\cos 2\alpha + \cos 2\beta) x + \cos 2\alpha \cos 2\beta = 0.$$

$$\text{Now } \cos 2\alpha + \cos 2\beta = 2 \cos(\alpha + \beta) \cos(\alpha - \beta)$$

$$= \frac{2(a^2 - b^2)}{a^2 + b^2} \cdot \frac{2c^2 - a^2 - b^2}{a^2 + b^2}.$$

[See solutions to examples 4 and 6 above.]

$$\text{Again, } \cos 2\alpha \cos 2\beta = \cos^2(\alpha - \beta) - \sin^2(\alpha + \beta)$$

$$= \frac{(2c^2 - a^2 - b^2)^2 - 4a^2b^2}{(a^2 + b^2)^2}.$$

Hence, by substitution the required equation is obtained.

Otherwise. Let $\cos \alpha = y$, then the given equation may be written

$$(ay - c)^2 = b^2(1 - y^2),$$

$$\text{If } x = \cos 2\alpha = 2 \cos^2 \alpha - 1 = 2y^2 - 1,$$

$$y^2 = \frac{x+1}{2}.$$

Substituting in (1) we obtain the equation

$$\left\{ (a^2 + b^2) \frac{x+1}{2} - b^2 + c^2 \right\}^2 = 4a^2c^2 \cdot \left(\frac{x+1}{2} \right),$$

which reduces to

$$(a^2 + b^2)^2 x^2 - 2(a^2 - b^2)(2c^2 - a^2 - b^2)x + (a^4 + b^4 + 4c^4 - 2a^2b^2 - 4a^2c^2 - 4b^2c^2) = 0$$

EXAMPLES. XXIV. b. PAGE 307.

$$\begin{aligned} 1. \quad \Sigma \sin(\alpha - \theta) \sin(\beta - \gamma) &= \Sigma (\sin \alpha \cos \theta - \cos \alpha \sin \theta) \sin(\beta - \gamma) \\ &= \cos \theta \Sigma \sin \alpha \sin(\beta - \gamma) - \sin \theta \Sigma \cos \alpha \sin(\beta - \gamma) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 2. \quad \Sigma (\cos \beta \cos \gamma - \sin \beta \sin \gamma) \sin(\beta - \gamma) &= \Sigma \cos(\beta + \gamma) \sin(\beta - \gamma) \\ &= \frac{1}{2} \Sigma (\sin 2\beta - \sin 2\gamma) \\ &= 0. \end{aligned}$$

$$\therefore \Sigma \cos \beta \cos \gamma \sin(\beta - \gamma) = \Sigma \sin \beta \sin \gamma \sin(\beta - \gamma).$$

$$\begin{aligned} 3. \quad \Sigma \sin(\beta - \gamma) \cos(\beta + \gamma + \theta) &= \Sigma \sin(\beta - \gamma) \{ \cos \theta \cos(\beta + \gamma) - \sin \theta \sin(\beta + \gamma) \} \\ &= \cos \theta \Sigma \sin(\beta - \gamma) \cos(\beta + \gamma) - \sin \theta \Sigma \sin(\beta - \gamma) \sin(\beta + \gamma) \\ &= \frac{1}{2} \cos \theta \Sigma (\sin 2\beta - \sin 2\gamma) - \frac{1}{2} \sin \theta \Sigma (\cos 2\gamma - \cos 2\beta) \\ &= 0. \end{aligned}$$

$$\begin{aligned} 4. \quad \cos 2(\beta - \gamma) + \cos 2(\gamma - \alpha) + \cos 2(\alpha - \beta) &= 2 \cos(\beta - \alpha) \cos(\alpha + \beta - 2\gamma) + 2 \cos^2(\alpha - \beta) - 1 \\ &= 2 \cos(\alpha - \beta) \{ \cos(\alpha + \beta - 2\gamma) + \cos(\alpha - \beta) \} - 1 \\ &= 4 \cos(\alpha - \beta) \cos(\alpha - \gamma) \cos(\beta - \gamma) - 1. \end{aligned}$$

$$\begin{aligned} 5. \quad \Sigma \sin \beta \sin \gamma \sin(\beta - \gamma) &= \frac{1}{2} \Sigma \{ \cos(\beta - \gamma) - \cos(\beta + \gamma) \} \sin(\beta - \gamma) \\ &= \frac{1}{4} \Sigma \sin 2(\beta - \gamma) - \frac{1}{4} \Sigma (\sin 2\beta - \sin 2\gamma) \\ &= -\text{II} \sin(\beta - \gamma). \end{aligned}$$

[Art. 306, Ex. 2.]

$$\begin{aligned} 6. \quad \cot(\alpha - \beta) &= \cot \{ (\alpha - \gamma) - (\beta - \gamma) \} \\ &= \frac{\cot(\alpha - \gamma) \cot(\beta - \gamma) + 1}{\cot(\beta - \gamma) - \cot(\alpha - \gamma)}; \end{aligned}$$

by multiplying up and transposing we obtain the required result.

$$7. \quad 2\Sigma \sin 3\alpha \sin (\beta - \gamma) = \cos (3\alpha - \beta + \gamma) - \cos (3\alpha + \beta - \gamma) + \cos (3\beta - \gamma + \alpha) \\ - \cos (3\beta + \gamma - \alpha) + \cos (3\gamma - \alpha + \beta) - \cos (3\gamma + \alpha - \beta).$$

Combining the first and fourth terms, the second and fifth terms, the third and sixth terms, and dividing by 2, we obtain

$$\Sigma \sin 3\alpha \sin (\beta - \gamma) = \sin (\alpha + \beta + \gamma) \{ \sin 2(\beta - \alpha) + \sin 2(\alpha - \gamma) + \sin 2(\gamma - \beta) \} \\ = 4 \sin (\alpha + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 2.}]$$

$$8. \quad 4\Sigma \cos^3 \alpha \sin (\beta - \gamma) := \Sigma (\cos 3\alpha + 3 \cos \alpha) \sin (\beta - \gamma) \\ = \Sigma \cos 3\alpha \sin (\beta - \gamma) \\ = 4 \cos (\alpha + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 4.}]$$

$$9. \quad 4\Sigma \cos (\theta + \alpha) \sin (\theta - \alpha) \cos (\beta + \gamma) \sin (\beta - \gamma) \\ = \Sigma (\sin 2\theta - \sin 2\alpha) (\sin 2\beta - \sin 2\gamma) \\ = \sin 2\theta \Sigma (\sin 2\beta - \sin 2\gamma) - \Sigma \sin 2\alpha (\sin 2\beta - \sin 2\gamma) \\ = 0.$$

10. In the identity $\Sigma bc(b - c) = - \Pi(b - c)$, put

$$a = \sin^2 \alpha, \quad b = \sin^2 \beta, \quad c = \sin^2 \gamma;$$

$$\text{then } b - c = \sin^2 \beta - \sin^2 \gamma = \sin (\beta + \gamma) \sin (\beta - \gamma);$$

$$\therefore \Sigma \sin^2 \beta \sin^2 \gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = - \Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

11. In the identity $\Sigma bc(b - c) = - \Pi(b - c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

$$\text{then } b - c = \cos 2\beta - \cos 2\gamma = - 2 \sin (\beta + \gamma) \sin (\beta - \gamma);$$

$$\therefore - 2\Sigma \cos 2\beta \cos 2\gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

12. In the identity $\Sigma a^2(b - c) = - \Pi(b - c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

$$\text{then } b - c = \cos 2\beta - \cos 2\gamma = - 2 \sin (\beta + \gamma) \sin (\beta - \gamma);$$

$$\therefore \Sigma 2 \cos^2 2\alpha \sin (\beta + \gamma) \sin (\beta - \gamma) = - 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

$$\text{Also } \Sigma \sin (\beta + \gamma) \sin (\beta - \gamma) = 0;$$

whence by subtraction, we have

$$\Sigma (2 \cos^2 2\alpha - 1) \sin (\beta + \gamma) \sin (\beta - \gamma) = - 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

13. In the identity $\Sigma a^3(b-c) = -(a+b+c)\Pi(b-c)$, put

$$a=\sin \alpha, \quad b=\sin \beta, \quad c=\sin \gamma;$$

$$\therefore \Sigma \sin^3 \alpha (\sin \beta - \sin \gamma) = -(\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma) \dots (1).$$

$$\text{But} \quad \Sigma \sin \alpha (\sin \beta - \sin \gamma) = 0 \dots (2);$$

multiply (2) by 3, and (1) by 4; then by subtraction we obtain

$$\Sigma (3 \sin \alpha - 4 \sin^3 \alpha) (\sin \beta - \sin \gamma) = 4 (\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma).$$

14. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

The condition $a+b+c=0$ is satisfied, if $a=\sin(\beta+\gamma)\sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

15. The condition $a+b+c=0$ is satisfied, if $a=\cos(\beta+\gamma+\theta)\sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

16. We proceed exactly as in Art. 309, and shew that

$$\alpha + \beta + \gamma = n\pi;$$

$$\therefore 3\alpha + 3\beta + 3\gamma = 3n\pi.$$

From this relation it is easy to shew that

$$\Sigma \tan 3\alpha = \Pi \tan 3\alpha;$$

$$\therefore \Sigma \frac{3x - x^3}{1 - 3x^2} = \Pi \frac{3x - x^3}{1 - 3x^2}.$$

17. Put $x=\cot \alpha, y=\cot \beta, z=\cot \gamma$; then

$$\cot \beta \cot \gamma + \cot \gamma \cot \alpha + \cot \alpha \cot \beta = 1;$$

$$\therefore \cot \alpha = -\frac{\cot \beta \cot \gamma - 1}{\cot \gamma + \cot \beta} = -\cot(\beta + \gamma);$$

$$\therefore \alpha = n\pi - (\beta + \gamma), \text{ or } \alpha + \beta + \gamma = n\pi;$$

$$\therefore 2\alpha + 2\beta + 2\gamma = 2n\pi.$$

From this relation it is easy to shew that

$$\cot 2\beta \cot 2\gamma + \cot 2\gamma \cot 2\alpha + \cot 2\alpha \cot 2\beta = 1;$$

$$\therefore \frac{(y^2 - 1)(z^2 - 1)}{4yz} + \frac{(z^2 - 1)(x^2 - 1)}{4zx} + \frac{(x^2 - 1)(y^2 - 1)}{4xy} = 1,$$

$$\therefore \Sigma x(1-y^2)(1-z^2) = 4xyz.$$

EXAMPLES. XXIV. c. PAGE 311.

1. If

$$A+B+C=0,$$

then $\cot C = -\cot(A+B) = -\frac{\cot A \cot B - 1}{\cot B + \cot A};$ that is, $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$

The given condition is satisfied if

$$A=2\beta+\gamma-3\alpha, \quad B=2\gamma+\alpha-3\beta, \quad C=2\alpha+\beta-3\gamma.$$

2. (1) In Example 4, Art. 133, we have proved that

$$4 \sin \alpha \sin \beta \sin \gamma = \Sigma \sin(\beta + \gamma - \alpha) - \sin(\alpha + \beta + \gamma).$$

In this identity first replace α, β, γ by $\beta + \gamma, \gamma + \alpha, \alpha + \beta$ respectively, and secondly replace α, β, γ by $2\alpha, 2\beta, 2\gamma$ respectively.Thus $8\Pi \sin(\beta + \gamma) = 2 \sin 2\alpha - 2 \sin 2(\alpha + \beta + \gamma),$ and $4\Pi \sin 2\alpha = \Sigma \sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma).$

$$\begin{aligned} \therefore 8\Pi \sin(\beta + \gamma) + 4\Pi \sin 2\alpha &= 2\Sigma \sin 2\alpha + \Sigma \sin 2(\beta + \gamma - \alpha) - 3 \sin 2(\alpha + \beta + \gamma) \\ &= 2\Sigma \sin 2\alpha + \Sigma \{ \sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma) \} \\ &= 2\Sigma \sin 2\alpha - 2\Sigma \cos 2(\beta + \gamma) \sin 2\alpha \\ &= 2\Sigma \sin 2\alpha \{ 1 - \cos 2(\beta + \gamma) \} \\ &= 4\Sigma \sin 2\alpha \sin^2(\beta + \gamma). \end{aligned}$$

(2) In the first part of this Example, we have seen that

$$8\Pi \sin \alpha = 2\Sigma \sin(\beta + \gamma - \alpha) - 2 \sin(\alpha + \beta + \gamma).$$

By replacing α, β, γ by $\beta + \gamma - \alpha, \gamma + \alpha - \beta, \alpha + \beta - \gamma$ respectively, we have

$$4\Pi \sin(\beta + \gamma - \alpha) = \Sigma \sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma).$$

$$\begin{aligned} \therefore 4\Pi \sin(\beta + \gamma - \alpha) + 8\Pi \sin \alpha &= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \sin(3\alpha - \beta - \gamma) - 3 \sin(\alpha + \beta + \gamma) \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \{ \sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma) \} \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) - 2\Sigma \cos 2\alpha \sin(\beta + \gamma - \alpha) \\ &= 2\Sigma \sin(\beta + \gamma - \alpha) \{ 1 - \cos 2\alpha \} \\ &= 4\Sigma \sin^2 \alpha \sin(\beta + \gamma - \alpha). \end{aligned}$$

3. (1) This is equivalent to proving that in the pedal triangle

$$a'^2 - b'^2 = 2R'c' \sin(A' - B').$$

Now in *any* triangle

$$\begin{aligned} a^2 - b^2 &= 4R^2(\sin^2 A - \sin^2 B) \\ &= 4R^2 \sin(A+B) \sin(A-B) \\ &= 2R \cdot 2R \sin C \cdot \sin(A-B) \\ &= 2Rc \sin(A-B). \end{aligned}$$

(2) This is equivalent to proving that in the ex-central triangle

$$a_1^2 - b_1^2 = 2R_1c_1 \sin(A_1 - B_1).$$

This identity has been proved in (1).

(3) This is equivalent to proving that in the pedal triangle

$$\Sigma(b' + c') \tan \frac{A'}{2} = 4R'\Sigma \cos A'.$$

Now in *any* triangle

$$\begin{aligned} \Sigma(b+c) \tan \frac{A}{2} &= 2R\Sigma(\sin B + \sin C) \tan \frac{A}{2} \\ &= 4R\Sigma \sin \frac{B+C}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \div \cos \frac{A}{2} \\ &= 4R\Sigma \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\ &= 2R\Sigma(\cos B + \cos C) \\ &= 4R(\cos A + \cos B + \cos C). \end{aligned}$$

4. We have $\sin^2 2\theta = 4 \sin^2 \alpha \sin^2 \gamma = (1 - \cos 2\alpha)(1 - \cos 2\gamma)$;

$$\therefore 1 - \cos^2 2\theta = \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right)\left(1 - \frac{\cos 2\theta}{\cos 2\delta}\right);$$

$$\therefore \cos 2\theta \left(\frac{1}{\cos 2\beta} + \frac{1}{\cos 2\delta}\right) = \cos^2 2\theta \left(1 + \frac{1}{\cos 2\beta \cos 2\delta}\right);$$

$$\therefore \cos 2\theta = \frac{\cos 2\beta + \cos 2\delta}{1 + \cos 2\beta \cos 2\delta};$$

$$\therefore \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{(1 - \cos 2\beta)(1 - \cos 2\delta)}{(1 + \cos 2\beta)(1 + \cos 2\delta)};$$

$$\therefore \tan^2 \theta = \tan^2 \beta \tan^2 \delta.$$

5. We have

$$\tan^2 \frac{\gamma}{2} = \tan^2 \frac{\theta}{2} \tan^2 \frac{\phi}{2};$$

$$\begin{aligned}\therefore \frac{1 - \cos \gamma}{1 + \cos \gamma} &= \frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{1 - \cos \phi}{1 + \cos \phi} \\ &= \frac{1 - \cos \alpha \cos \gamma}{1 + \cos \alpha \cos \gamma} \cdot \frac{1 - \cos \beta \cos \gamma}{1 + \cos \beta \cos \gamma}.\end{aligned}$$

Componendo and Dividendo,

$$\frac{1}{\cos \gamma} = \frac{1 + \cos \alpha \cos \beta \cos^2 \gamma}{(\cos \alpha + \cos \beta) \cos \gamma};$$

$$\therefore \cos^2 \gamma = \frac{\cos \alpha + \cos \beta - 1}{\cos \alpha \cos \beta};$$

$$\therefore \sin^2 \gamma = \frac{(1 - \cos \alpha)(1 - \cos \beta)}{\cos \alpha \cos \beta} = (\sec \alpha - 1)(\sec \beta - 1).$$

6. By solving for $\cos \theta$, we have

$$\cos \theta = \frac{\sin^2 \beta \cos^2 \alpha - \sin^2 \alpha \cos^2 \beta}{\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta}.$$

By substituting $1 - \cos^2 \beta$ for $\sin^2 \beta$, and $1 - \cos^2 \alpha$ for $\sin^2 \alpha$, we have

$$\begin{aligned}\cos \theta &= \frac{\cos^2 \alpha - \cos^2 \beta}{(1 - \cos^2 \beta) \cos \alpha - (1 - \cos^2 \alpha) \cos \beta} \\ &= \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}.\end{aligned}$$

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{(1 - \cos \alpha)(1 - \cos \beta)}{(1 + \cos \alpha)(1 + \cos \beta)};$$

$$\therefore \tan^2 \frac{\theta}{2} = \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2}.$$

7. From the two given equations, we have

$$\sec \alpha = \frac{\tan \beta}{\tan \theta}, \text{ and } \tan \alpha = \frac{\tan \gamma}{\sin \theta};$$

$$\therefore \frac{\tan^2 \beta}{\tan^2 \theta} - \frac{\tan^2 \gamma}{\sin^2 \theta} = 1;$$

$$\therefore \frac{\tan^2 \beta \cos^2 \theta}{1 - \cos^2 \theta} - \frac{\tan^2 \gamma}{1 - \cos^2 \theta} = 1;$$

$$\therefore \tan^2 \beta \cos^2 \theta - \tan^2 \gamma = 1 - \cos^2 \theta;$$

$$\therefore \cos^2 \theta = \frac{1 + \tan^2 \gamma}{1 + \tan^2 \beta} = \frac{\sec^2 \gamma}{\sec^2 \beta}.$$

$$8. \text{ We have } bc \cos \alpha \cos \phi + ac \sin \alpha \sin \phi - ab = 0,$$

whence by cross multiplication

$$\begin{aligned} \frac{\cos \phi}{a^2bc(\sin \beta - \sin \alpha)} &= \frac{\sin \phi}{ab^2c(\cos \alpha - \cos \beta)} = \frac{1}{abc^2 \sin(\beta - \alpha)}; \\ \therefore \frac{\cos \phi}{a \cos \frac{\alpha + \beta}{2}} &= \frac{\sin \phi}{b \sin \frac{\alpha + \beta}{2}} = \frac{1}{c \cos \frac{\alpha - \beta}{2}}; \\ \therefore a^2 \cos^2 \frac{\alpha + \beta}{2} + b^2 \sin^2 \frac{\alpha + \beta}{2} &= c^2 \cos^2 \frac{\alpha - \beta}{2}; \end{aligned}$$

$$\therefore (b^2 + c^2 - a^2) \cos \alpha \cos \beta + (c^2 + a^2 - b^2) \sin \alpha \sin \beta = a^2 + b^2 - c^2.$$

9. From the given equation, we have

$$\sin^2 \alpha (\cos \theta - \cos \alpha)^2 = \cos^2 \alpha \sin^2 \theta = \cos^2 \alpha (1 - \cos^2 \theta);$$

$$\therefore \cos^2 \theta - 2 \cos \alpha \sin^2 \alpha \cos \theta - \cos^4 \alpha = 0;$$

which is a quadratic in $\cos \theta$ with roots $\cos \beta$ and $\cos \gamma$.

$$\therefore \cos \beta \cos \gamma = \cos^4 a.$$

Similarly, from the equation

$$\cos^2 \alpha (\sin \theta - \sin \alpha)^2 = \sin^2 \alpha \cos^2 \theta = \sin^2 \alpha (1 - \sin^2 \theta),$$

$$\text{we may shew that } \sin \beta \sin \gamma = \sin^4 a.$$

$$\therefore \frac{\cos \beta \cos \gamma}{\cos^2 \alpha} + \frac{\sin \beta \sin \gamma}{\sin^2 \alpha} = \cos^2 \alpha + \sin^2 \alpha = 1.$$

10. We have $k^2 \cos \beta \cos \alpha + k \sin \alpha + (k \sin \beta + 1) = 0$ (1),
 and $k^2 \cos \gamma \cos \alpha + k \sin \alpha + (k \sin \gamma + 1) = 0$ (2);

whence by cross multiplication,

$$\frac{\cos \alpha}{k^2(\sin \beta - \sin \gamma)} = \frac{\sin \alpha}{k^3 \sin(\beta - \gamma) + k^2(\cos \gamma - \cos \beta)} = \frac{1}{k^3(\cos \beta - \cos \gamma)};$$

$$\therefore \frac{\cos \alpha}{\cos \frac{\beta+\gamma}{2}} = \frac{\sin \alpha}{k \cos \frac{\beta-\gamma}{2} + \sin \frac{\beta+\gamma}{2}} = - \frac{1}{k \sin \frac{\beta+\gamma}{2}};$$

$$\therefore \cos^2 \frac{\beta+\gamma}{2} + \left\{ k \cos \frac{\beta-\gamma}{2} + \sin \frac{\beta+\gamma}{2} \right\}^2 = k^2 \sin^2 \frac{\beta+\gamma}{2};$$

$$\therefore k^2 \left(\cos^2 \frac{\beta - \gamma}{2} - \sin^2 \frac{\beta + \gamma}{2} \right) + k (\sin \beta + \sin \gamma) + 1 = 0;$$

$$\therefore k^2 \cos \beta \cos \gamma + k (\sin \beta + \sin \gamma) + 1 = 0.$$

Otherwise. Form a quadratic equation in $\sin \theta$; thus

$$k^4 \cos^2 \alpha (1 - \sin^2 \theta) = (1 + k \sin \alpha + k \sin \theta)^2;$$

$\sin \beta + \sin \gamma$ = sum of roots of this equation

$$= -\frac{\text{coefficient of } \sin \theta}{\text{coefficient of } \sin^2 \theta}$$

$$= -\frac{2(1+k \sin \alpha)k}{k^2(1+k^2 \cos^2 \alpha)};$$

$$\therefore k(\sin \beta + \sin \gamma) = -\frac{2(1+k \sin \alpha)}{1+k^2 \cos^2 \alpha} \dots \dots \dots (1).$$

Again, form a quadratic equation in $\cos \theta$; thus

$$k^2(1 - \cos^2 \theta) = (1 + k \sin \alpha + k^2 \cos \alpha \cos \theta)^2;$$

$\cos \beta \cos \gamma$ = product of roots of this equation

$$= \frac{k^2 - (1+k \sin \alpha)^2}{-k^2 - k^4 \cos^2 \alpha};$$

$$\therefore k^2 \cos \beta \cos \gamma = \frac{1+2k \sin \alpha - k^2 \cos^2 \alpha}{1+k^2 \cos^2 \alpha} \dots \dots \dots (2).$$

By adding (1) and (2) we obtain the required result.

11. We have

$$\sin^2 \alpha \cos \beta \cos \phi + \cos^2 \alpha \sin \beta \sin \phi + \cos^2 \alpha \sin^2 \alpha = 0,$$

$$\text{and } \sin^2 \alpha \cos \gamma \cos \phi + \cos^2 \alpha \sin \gamma \sin \phi + \cos^2 \alpha \sin^2 \alpha = 0;$$

whence by cross multiplication

$$\frac{\cos \phi}{\cos^4 \alpha \sin^2 \alpha (\sin \beta - \sin \gamma)} = \frac{\sin \phi}{\cos^2 \alpha \sin^4 \alpha (\cos \gamma - \cos \beta)}$$

$$= \frac{1}{\sin^2 \alpha \cos^2 \alpha \sin(\gamma - \beta)};$$

$$\therefore \frac{\cos \phi}{\cos^2 \alpha \cos \frac{\beta+\gamma}{2}} = \frac{\sin \phi}{\sin^2 \alpha \sin \frac{\beta+\gamma}{2}} = -\frac{1}{\cos \frac{\beta-\gamma}{2}};$$

$$\therefore \cos^4 \alpha \cos^2 \frac{\beta+\gamma}{2} + \sin^4 \alpha \sin^2 \frac{\beta+\gamma}{2} = \cos^2 \frac{\beta-\gamma}{2};$$

$$\therefore \cos^4 \alpha \{1 + \cos(\beta + \gamma)\} + \sin^4 \alpha \{1 - \cos(\beta + \gamma)\} = 1 + \cos(\beta - \gamma);$$

$$\therefore \cos \beta \cos \gamma (1 - \cos^4 \alpha + \sin^4 \alpha) + \sin \beta \sin \gamma (1 + \cos^4 \alpha - \sin^4 \alpha)$$

$$= \cos^4 \alpha + \sin^4 \alpha - 1.$$

Writing $\cos^4 \alpha + \sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha$ instead of unity we have

$$2 \cos \beta \cos \gamma \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha) + 2 \sin \beta \sin \gamma \cos^2 \alpha (\cos^2 \alpha + \sin^2 \alpha)$$

$$= -2 \sin^2 \alpha \cos^2 \alpha;$$

$$\text{that is, } \sin^2 \alpha \cos \beta \cos \gamma + \cos^2 \alpha \sin \beta \sin \gamma + \sin^2 \alpha \cos^2 \alpha = 0.$$

EXAMPLES. XXV. a. PAGE 318.

$$1. \quad p \cot \theta + q \tan \theta = (\sqrt{p \cot \theta} - \sqrt{q \tan \theta})^2 + 2 \sqrt{pq}.$$

$$2. \quad 4 \sin^2 \theta + \operatorname{cosec}^2 \theta = (2 \sin \theta - \operatorname{cosec} \theta)^2 + 4.$$

$$3. \quad 8 \sec^2 \theta + 18 \cos^2 \theta = 2 \{(2 \sec \theta - 3 \cos \theta)^2 + 12\}.$$

$$4. \quad 3 - 2 \cos \theta + \cos^2 \theta = 2 + (1 - \cos \theta)^2.$$

5. We have $\tan^2 \beta + \tan^2 \gamma > 2 \tan \beta \tan \gamma$, and two corresponding inequalities. [See Art. 316.]

6. Since $(1 - \sin \alpha)^2$ is positive, $1 + \sin^2 \alpha > 2 \sin \alpha$.

$$\text{Similarly,} \quad 1 + \sin^2 \beta > 2 \sin \beta.$$

$$\therefore 2 + \sin^2 \alpha + \sin^2 \beta > 2 \sin \alpha + 2 \sin \beta.$$

$$7. \quad \sin \theta + \cos \theta = \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right).$$

$$8. \quad \cos \theta + \sqrt{3} \sin \theta = 2 \sin \left(\theta + \frac{\pi}{6} \right).$$

$$9. \quad a \cos(\alpha + \theta) + b \sin \theta = a \cos \alpha \cos \theta + (b - a \sin \alpha) \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{a^2 \cos^2 \alpha + (b - a \sin \alpha)^2}. \quad [\text{Art. 317.}]$$

$$10. \quad p \cos \theta + q \sin(\alpha + \theta) = (p + q \sin \alpha) \cos \theta + q \cos \alpha \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{(p + q \sin \alpha)^2 + q^2 \cos^2 \alpha}.$$

$$11. \quad \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = 2 \sin \frac{\sigma}{2} \cos \frac{\alpha - \beta}{2}.$$

$$\therefore \text{maximum value} = 2 \sin \frac{\sigma}{2}.$$

$$12. \quad \sin \alpha \sin \beta = \frac{1}{2} \{ \cos(\alpha - \beta) - \cos(\alpha + \beta) \} = \frac{1}{2} \{ \cos(\alpha - \beta) - \cos \sigma \}.$$

$$\therefore \text{maximum value} = \frac{1}{2} (1 - \cos \sigma) = \sin^2 \frac{\sigma}{2}.$$

$$13. \quad \tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \frac{\sin \sigma}{\cos \alpha \cos \beta}.$$

By Art. 319, the denominator is a maximum when $\alpha = \beta$, and in this case $\tan \alpha + \tan \beta$ is a minimum, its value being $2 \tan \frac{\sigma}{2}$.

$$\begin{aligned}
 14. \quad & \text{cosec } \alpha + \text{cosec } \beta = \frac{\sin \alpha + \sin \beta}{\sin \alpha \sin \beta} \\
 & = \frac{4 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}{\cos(\alpha-\beta) - \cos(\alpha+\beta)} = \frac{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}{\cos^2 \frac{\alpha-\beta}{2} - \cos^2 \frac{\alpha+\beta}{2}} \\
 & = \sin \frac{\alpha+\beta}{2} \left(\frac{1}{\cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2}} + \frac{1}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}} \right).
 \end{aligned}$$

Since $\alpha+\beta$ is constant, this expression is least when the denominators are greatest, that is, when $\alpha=\beta=\frac{\sigma}{2}$.

Thus the minimum value is $2 \operatorname{cosec} \frac{\sigma}{2}$.

$$\begin{aligned}
 15. \quad & \cos A \cos B \cos C = \frac{1}{2} \cos C \{ \cos(A-B) + \cos(A+B) \} \\
 & \quad = \frac{1}{2} \cos C \{ \cos(A-B) - \cos C \}.
 \end{aligned}$$

Supposing C constant, this expression is not a maximum unless $A=B$. Similarly, we may shew that the given expression is not a maximum unless $A=B=C=60^\circ$. In this case its value is $\cos^3 60^\circ$ or $\frac{1}{8}$.

$$16. \quad \cot A + \cot B + \cot C = \frac{\sin(A+B)}{\sin A \sin B} + \cot C = \frac{\sin C}{\sin A \sin B} + \cot C.$$

Supposing C constant, $\sin A \sin B$ is a maximum when $A=B$, and in this case the given expression is a minimum. Thus the given expression is a minimum when $A=B=C$, and its value is $3 \cot 60^\circ$ or $\sqrt{3}$.

$$\begin{aligned}
 17. \quad & \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = \frac{1}{2} (1 - \cos A) + \dots + \dots \\
 & \quad = \frac{3}{2} - \frac{1}{2} (\cos A + \cos B + \cos C).
 \end{aligned}$$

As in Example 1, p. 315, it is easy to shew that $\cos A + \cos B + \cos C$ is a maximum when $A=B=C$, and in this case the given expression is a minimum, its value being $3 \sin^2 \frac{60^\circ}{2}$ or $\frac{3}{4}$.

18. If C is constant, $A+B$ is constant, and therefore $\sec A + \sec B$ is a minimum when $A=B$. [See Ex. 2, p. 316.]

Thus the given expression is a minimum when $A=B=C$, its value being $3 \sec 60^\circ$ or 6.

$$\begin{aligned}
 19. \quad & \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \\
 &= \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2 \Sigma \tan \frac{B}{2} \tan \frac{C}{2} \\
 &= \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2.
 \end{aligned}$$

As in Example 16, it is easy to shew that $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}$ is a minimum when $A = B = C$, and in this case the given expression is a minimum, its value being $3 \tan^2 \frac{60^\circ}{2}$ or 1.

$$\begin{aligned}
 20. \quad & \cot^2 A + \cot^2 B + \cot^2 C = (\cot A + \cot B + \cot C)^2 - 2 \Sigma \cot B \cot C \\
 &= (\cot A + \cot B + \cot C)^2 - 2.
 \end{aligned}$$

But it has been shewn in Example 16 that the right side is a minimum when $A = B = C$, and in this case the given expression is a minimum, its value being $3 \cot^2 60^\circ$ or 1.

$$\begin{aligned}
 21. \quad & 2(a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta) \\
 &= a(1 - \cos 2\theta) + b \sin 2\theta + c(1 + \cos 2\theta) \\
 &= a + c + b \sin 2\theta - (a - c) \cos 2\theta \quad \dots \dots \dots (1).
 \end{aligned}$$

The greatest value of $b \sin 2\theta - (a - c) \cos 2\theta$ is $\sqrt{b^2 + (a - c)^2}$, which is less than $\sqrt{4ac + (a - c)^2}$ or $a + c$, since $b^2 < 4ac$.

Hence as in Art. 317, the maximum and minimum values of (1) are

$$a + c \pm \sqrt{b^2 + (a - c)^2}.$$

$$\begin{aligned}
 22. \quad & \sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{\alpha + \beta}{2} \\
 &= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + \alpha}{2} \sin \frac{\beta + \gamma}{2}.
 \end{aligned}$$

The expression on the right is positive, since each of its component factors is positive.

$$\therefore \sin \alpha + \sin \beta + \sin \gamma > \sin(\alpha + \beta + \gamma).$$

23. Let $x = a \operatorname{cosec} \theta - b \cot \theta$, and put $\cot \theta = t$;

then $x = a \sqrt{1 + t^2} - bt$.

As in Ex. 2, page 317, we may shew that $x > \sqrt{a^2 - b^2}$;

$$\therefore a \operatorname{cosec} \theta - b \cot \theta > \sqrt{a^2 - b^2}.$$

24. Let $x = \frac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta} = \frac{1 - \tan \theta + \tan^2 \theta}{1 + \tan \theta + \tan^2 \theta};$

$$\therefore \tan^2 \theta (x - 1) + \tan \theta (x + 1) + x - 1 = 0.$$

In order that the values of $\tan \theta$ found from this quadratic may be real, we must have

$$(x + 1)^2 > 4(x - 1)^2;$$

that is,

$$-3x^2 + 10x - 3 \text{ must be positive;} \\$$

that is,

$$(3x - 1)(x - 3) \text{ must be negative.}$$

Hence x must lie between 3 and $\frac{1}{3}$.

25. Denote the expression by x ; then

$$x = \frac{\tan^4 \theta + \tan^2 \theta - 1}{\tan^4 \theta - \tan^2 \theta + 1};$$

$$\therefore \tan^4 \theta (x - 1) - \tan^2 \theta (x + 1) + x + 1 = 0.$$

In order that the values of $\tan^2 \theta$ found from this equation may be real, we must have

$$(x + 1)^2 > 4(x + 1)(x - 1);$$

$$\therefore (x + 1)(5 - 3x) > 0.$$

Thus the greatest value of x is $\frac{5}{3}$.

26. We have

$$(a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2 \\ = (b \cos \gamma - c \cos \beta)^2 + (c \cos \alpha - a \cos \gamma)^2 + (a \cos \beta - b \cos \alpha)^2,$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - k^2 = 0.$$

$$\text{Again, } (a + b + c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2$$

$$= bc(\cos \beta - \cos \gamma)^2 + ca(\cos \gamma - \cos \alpha)^2 + ab(\cos \alpha - \cos \beta)^2,$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a + b + c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - k^2 = 0.$$

EXAMPLES. XXV. b. PAGE 324.

1. By squaring each of the given equations and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

2. By transposition, we have

$$a \sec \theta = x \tan \theta + y, \quad b \sec \theta = x - y \tan \theta;$$

whence by squaring and adding,

$$(a^2 + b^2) \sec^2 \theta = (x^2 + y^2)(1 + \tan^2 \theta),$$

or

$$x^2 + y^2 = a^2 + b^2.$$

3. We have $\cos \theta + \sin \theta = a$, and $\cos^2 \theta - \sin^2 \theta = b$;

$$\therefore \cos \theta - \sin \theta = \frac{b}{a};$$

$$\therefore a^2 + \frac{b^2}{a^2} = 2.$$

4. We have $x^2 = (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$,

and $y = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{2}{\sin 2\theta};$

$$\therefore y(x^2 - 1) = 2.$$

5. By addition and subtraction, we have

$$\cot \theta = \frac{a+b}{2}, \text{ and } \cos \theta = \frac{a-b}{2};$$

whence by division, $\sin \theta = \frac{a-b}{a+b};$

$$\therefore \left(\frac{a-b}{2}\right)^2 + \left(\frac{a-b}{a+b}\right)^2 = 1, \text{ or } \left(\frac{a-b}{2}\right)^2 = \frac{4ab}{(a+b)^2},$$

that is, $(a^2 - b^2)^2 = 16ab.$

6. Here $x = \cot \theta + \frac{1}{\cot \theta} = \frac{\cot^2 \theta + 1}{\cot \theta} = \frac{\operatorname{cosec}^2 \theta}{\cot \theta}$,

and $y = \operatorname{cosec} \theta - \frac{1}{\operatorname{cosec} \theta} = \frac{\operatorname{cosec}^2 \theta - 1}{\operatorname{cosec} \theta} = \frac{\cot^2 \theta}{\operatorname{cosec} \theta};$

$$\therefore x^2 y = \operatorname{cosec}^3 \theta, \text{ and } xy^2 = \cot^3 \theta.$$

But $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1;$

$$\therefore x^{\frac{4}{3}} y^{\frac{2}{3}} - x^{\frac{2}{3}} y^{\frac{4}{3}} = 1.$$

7. Here $a^3 = \frac{1}{\sin \theta} - \sin \theta = \frac{\cos^2 \theta}{\sin \theta}$,

and $b^3 = \frac{1}{\cos \theta} - \cos \theta = \frac{\sin^2 \theta}{\cos \theta}$;

$$\therefore a^6 b^3 = \cos^3 \theta, \text{ or } a^2 b = \cos \theta;$$

and $a^3 b^6 = \sin^3 \theta, \text{ or } a b^2 = \sin \theta;$

$$\therefore a^4 b^2 + a^2 b^4 = 1.$$

8. By substituting for $\cos 3\theta$ and $\sin 3\theta$, we have

$$4x = 4a \cos^3 \theta, \text{ or } x^{\frac{1}{3}} = a^{\frac{1}{3}} \cos \theta,$$

and $4y = 4a \sin^3 \theta, \text{ or } y^{\frac{1}{3}} = a^{\frac{1}{3}} \sin \theta;$

$$\therefore x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{9}{2}}.$$

9. By transposition, we have

$$x(1 + \tan^2 \theta) = a \tan^3 \theta, \text{ or } x = \frac{a \tan^3 \theta}{\sec^2 \theta},$$

and $y(\sec^2 \theta - 1) = a \sec^3 \theta, \text{ or } y = \frac{a \sec^3 \theta}{\tan^2 \theta};$

$$\therefore x^3 y^2 = a^5 \tan^5 \theta, \text{ and } x^2 y^3 = a^5 \sec^5 \theta.$$

$$\therefore (x^2 y^3)^{\frac{2}{5}} - (x^3 y^2)^{\frac{2}{5}} = a^2 (\sec^2 \theta - \tan^2 \theta) = a^2.$$

10. Here $x = a \cos \theta (2 \cos 2\theta - 1) = a \cos \theta (4 \cos^2 \theta - 3)$
 $= a \cos 3\theta.$

Similarly $y = b \sin 3\theta.$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

11. Here $\sin \theta \sin \alpha + \cos \theta \cos \alpha = a,$

and $\sin \theta \cos \beta - \cos \theta \sin \beta = b;$

$$\therefore \sin \theta \cos(\alpha - \beta) = a \sin \beta + b \cos \alpha,$$

and $\cos \theta \cos(\alpha - \beta) = a \cos \beta - b \sin \alpha;$

$$\therefore \cos^2(\alpha - \beta) = a^2 + b^2 - 2ab \sin(\alpha - \beta).$$

12. Here $x + y = 3 - (1 - 2 \sin^2 2\theta) = 2 + 2 \sin^2 2\theta,$

and $x - y = 4 \sin 2\theta;$

$$\therefore x = 1 + 2 \sin 2\theta + \sin^2 2\theta = (1 + \sin 2\theta)^2,$$

or $x^{\frac{1}{2}} = 1 + \sin 2\theta.$

Similarly, $y^{\frac{1}{2}} = 1 - \sin 2\theta.$

$$\therefore x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2.$$

13. By addition and subtraction, we have

$$\begin{aligned}x + y &= (\sin \theta + \cos \theta) (1 + \sin 2\theta) \\&= (\sin \theta + \cos \theta)^3,\end{aligned}$$

and

$$\begin{aligned}x - y &= (\sin \theta - \cos \theta) (1 - \sin 2\theta) \\&= (\sin \theta - \cos \theta)^3;\end{aligned}$$

$$\therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2.$$

14. Here

$$a^2 = 1 + \sin 2\theta,$$

$$\therefore a^2 - 1 + \cos 2\theta = b.$$

$$\therefore \sin 2\theta = a^2 - 1, \text{ and } \cos 2\theta = -(a^2 - b - 1);$$

$$\therefore (a^2 - 1)^2 + (a^2 - b - 1)^2 = 1.$$

15. Here

$$a = (4 \cos^3 \theta - 3 \cos \theta) + (3 \sin \theta - 4 \sin^3 \theta)$$

$$= (\cos \theta - \sin \theta) \{4(\cos^2 \theta + \cos \theta \sin \theta + \sin^2 \theta) - 3\}$$

$$= b(1 + 2 \sin 2\theta).$$

But

$$1 - \sin 2\theta = b^2;$$

$$\therefore a = b(1 + 2 - 2b^2) = 3b - 2b^3.$$

16. By squaring the first equation, and multiplying by 2, we have

$$a^2(1 + \cos 2\theta) + b^2(1 - \cos 2\theta) - 2ab \sin 2\theta = 2c^2;$$

$$\therefore (a^2 - b^2) \cos 2\theta - 2ab \sin 2\theta = 2c^2 - a^2 - b^2.$$

From the second equation,

$$(a^2 - b^2) \sin 2\theta + 2ab \cos 2\theta = 2c^2.$$

By squaring and adding, we obtain

$$\begin{aligned}(a^2 - b^2)^2 + 4a^2b^2 &= 4c^4 - 4c^2(a^2 + b^2) + (a^2 + b^2)^2 + 4c^4; \\ \therefore 0 &= 8c^4 - 4c^2(a^2 + b^2); \\ \therefore a^2 + b^2 &= 2c^2.\end{aligned}$$

17. By squaring and adding, we have

$$\begin{aligned}x^2 + y^2 &= a^2 + b^2 + 2ab(\cos \theta \cos 2\theta + \sin \theta \sin 2\theta) \\&= a^2 + b^2 + 2ab \cos \theta.\end{aligned}$$

Again,

$$x + b = a \cos \theta + 2b \cos^2 \theta = \cos \theta(a + 2b \cos \theta),$$

and

$$y = \sin \theta(a + 2b \cos \theta);$$

$$\therefore (x + b)^2 + y^2 = (a + 2b \cos \theta)^2$$

$$= \frac{1}{a^2} (x^2 + y^2 - b^2)^2.$$

18. Componendo and dividendo, we have

$$\frac{a}{b} = \frac{\tan(\theta + \alpha) + \tan(\theta - \alpha)}{\tan(\theta + \alpha) - \tan(\theta - \alpha)} = \frac{\sin 2\theta}{\sin 2\alpha};$$

$$\therefore b \sin 2\theta = a \sin 2\alpha.$$

Also

$$b \cos 2\theta = c - a \cos 2\alpha;$$

$$\therefore b^2 = c^2 - 2ac \cos 2\alpha + a^2.$$

19. By squaring and adding, we have

$$\begin{aligned} x^2 + y^2 &= a^2 \{2 - 2(\cos 3\theta \cos \theta + \sin 3\theta \sin \theta)\} \\ &= a^2 (2 - 2 \cos 2\theta) \\ &= 4a^2 \sin^2 \theta. \end{aligned}$$

And

$$2a^2 - x^2 - y^2 = 2a^2 \cos 2\theta.$$

But

$$\begin{aligned} x &= 2a \cos 2\theta \sin \theta; \\ \therefore 4a^4 x^2 &= (2a^2 \cos 2\theta)^2 (4a^2 \sin^2 \theta) \\ &= (2a^2 - x^2 - y^2)^2 (x^2 + y^2). \end{aligned}$$

20. Solving the given equations for x and y , we have

$$\begin{aligned} x &= a (\cos \theta \cos 2\theta + 2 \sin \theta \sin 2\theta); \\ \therefore 2x &= a \{(\cos 3\theta + \cos \theta) + 2(\cos \theta - \cos 3\theta)\} \\ &= a (3 \cos \theta - \cos 3\theta) = a (6 \cos \theta - 4 \cos^3 \theta). \end{aligned}$$

And

$$y = a (2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta);$$

$$\begin{aligned} \therefore 2y &= a \{2(\sin 3\theta + \sin \theta) - (\sin 3\theta - \sin \theta)\} \\ &= a (3 \sin \theta + \sin 3\theta) = a (6 \sin \theta - 4 \sin^3 \theta). \end{aligned}$$

$$\begin{aligned} \therefore 2(x+y) &= a (\cos \theta + \sin \theta) \{6 - 4(\cos^2 \theta - \cos \theta \sin \theta + \sin^2 \theta)\} \\ &= 2a (\cos \theta + \sin \theta) (1 + \sin 2\theta); \\ \therefore x+y &= a (\cos \theta + \sin \theta)^3. \end{aligned}$$

Similarly,

$$x-y = a (\cos \theta - \sin \theta)^3;$$

$$\therefore (x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

21. From the first equation,

$$(x \sin \theta - y \cos \theta)^2 = (x^2 + y^2) (\sin^2 \theta + \cos^2 \theta);$$

whence

$$x \cos \theta + y \sin \theta = 0;$$

$$\therefore \frac{\cos \theta}{y} = \frac{\sin \theta}{-x} = \frac{1}{\sqrt{x^2 + y^2}}.$$

By substituting in the second equation, we have

$$\frac{1}{x^2 + y^2} \left(\frac{y^2}{a^2} + \frac{x^2}{b^2} \right) = \frac{1}{x^2 + y^2};$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

22. From the first equation, we have

$$bx \cos \theta + ay \sin \theta = ab \sqrt{\sin^2 \theta + \cos^2 \theta};$$

$$\therefore b^2(x^2 - a^2) \cos^2 \theta + 2abxy \cos \theta \sin \theta + a^2(y^2 - b^2) \sin^2 \theta = 0.$$

From the second equation, we have

$$(y^2 - b^2) \cos^2 \theta - 2xy \cos \theta \sin \theta + (x^2 - a^2) \sin^2 \theta = 0.$$

Multiplying this equation by ab and adding to the preceding equation,

$$\{b^2(x^2 - a^2) + ab(y^2 - b^2)\} \cos^2 \theta + \{a^2(y^2 - b^2) + ab(x^2 - a^2)\} \sin^2 \theta = 0;$$

$$\therefore \{bx^2 + ay^2 - ab(a + b)\} (b \cos^2 \theta + a \sin^2 \theta) = 0.$$

$$\therefore bx^2 + ay^2 - ab(a + b) = 0,$$

or

$$b \cos^2 \theta + a \sin^2 \theta = 0.$$

From the last result,

$$\frac{\cos^2 \theta}{a} = \frac{\sin^2 \theta}{-b};$$

but

$$\{(y^2 - b^2) \cos^2 \theta + (x^2 - a^2) \sin^2 \theta\}^2 = 4x^2y^2 \cos^2 \theta \sin^2 \theta;$$

$$\therefore \{a(y^2 - b^2) - b(x^2 - a^2)\}^2 = -4abx^2y^2.$$

23. We have $4 \cos(\alpha - 3\theta) = m(3 \cos \theta + \cos 3\theta)$,

and

$$4 \sin(\alpha - 3\theta) = m(3 \sin \theta - \sin 3\theta);$$

whence by squaring and adding, we have

$$16 = m^2(10 + 6 \cos 4\theta).$$

Again, by multiplying the first equation by $\cos 3\theta$ and the second by $\sin 3\theta$ and subtracting, we obtain

$$4 \cos \alpha = m(3 \cos 4\theta + 1);$$

$$\therefore 16 = 10m^2 + 2m(4 \cos \alpha - m),$$

or

$$2 = m^2 + m \cos \alpha.$$

24. We have $\frac{x}{y} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi$,

But $\tan \alpha = \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$;

$$\therefore \tan \alpha = \frac{xy}{y - x}.$$

25. We have $a^2 + b^2 = 2 + 2 \cos(\theta - \phi)$;

$$\therefore a^2 + b^2 = 2 + 2 \cos \alpha.$$

26. We have $a \sin^2 \theta + b \cos^2 \theta = 1 = \sin^2 \theta + \cos^2 \theta$;

$$\therefore (a - 1) \tan^2 \theta = 1 - b.$$

Again,

$$a \cos^2 \phi + b \sin^2 \phi = 1 = \cos^2 \phi + \sin^2 \phi;$$

$$\therefore (b-1) \tan^2 \phi = 1 - a.$$

But

$$a^2 \tan^2 \theta = b^2 \tan^2 \phi;$$

$$\therefore \frac{a^2(1-b)}{a-1} = \frac{b^2(1-a)}{b-1};$$

$$\therefore a(b-1) = \pm b(a-1).$$

Rejecting the upper sign, we have $a+b=2ab$.

27. From the first two equations, we have

$$\frac{\frac{x}{a}}{\cos \frac{\theta+\phi}{2}} = \frac{\frac{y}{b}}{\sin \frac{\theta+\phi}{2}} = \frac{1}{\cos \frac{\theta-\phi}{2}};$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \frac{\theta-\phi}{2} = \sec^2 \frac{\alpha}{2}.$$

28. We have $\frac{a}{b} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi$.

But

$$\tan \alpha = \tan(\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi};$$

$$\therefore (a+b) \tan \alpha = b (\tan \theta - \tan \phi);$$

$$\begin{aligned} \therefore (a+b)^2 \tan^2 \alpha &= b^2 \{(\tan \theta + \tan \phi)^2 - 4 \tan \theta \tan \phi\} \\ &= b^2 a^2 - 4ab. \end{aligned}$$

29. We have $\frac{a \cos^2 \theta + b \sin^2 \theta}{a \sin^2 \theta + b \cos^2 \theta} = \frac{m \cos^2 \phi}{n \sin^2 \phi};$

$$\therefore \frac{a+b \tan^2 \theta}{a \tan^2 \theta + b} = \frac{m}{n \tan^2 \phi} = \frac{1}{\tan^2 \theta};$$

$$\therefore b \tan^4 \theta = b, \text{ or } \tan^2 \theta = \pm 1.$$

$$\therefore n \tan^2 \phi = \pm m.$$

By adding together the first two equations, we obtain

$$a+b = m \cos^2 \phi + n \sin^2 \phi.$$

If $n \tan^2 \phi = m$, then $\frac{\cos^2 \phi}{n} = \frac{\sin^2 \phi}{m} = \frac{1}{m+n};$

$$\therefore a+b = \frac{2mn}{m+n}.$$

If $n \tan^2 \phi = -m$, we obtain $a+b=0$.

30. From the second and third equations, we have by addition

$$2 \cos \phi (x \cos \theta + y \sin \theta) = 6a;$$

$$\therefore 2a\sqrt{3} \cos \phi = 3a, \text{ whence } \cos \phi = \frac{\sqrt{3}}{2}.$$

From the second and third equations, we have by subtraction

$$2 \sin \phi (-x \sin \theta + y \cos \theta) = 2a;$$

$$\therefore -x \sin \theta + y \cos \theta = \pm 2a.$$

But

$$x \cos \theta + y \sin \theta = 2a\sqrt{3};$$

$$\therefore x^2 + y^2 = 16a^2.$$

31. We have $c \sin \theta = a (\sin \theta \cos \phi + \cos \theta \sin \phi);$

$$\therefore (c - a \cos \phi) \sin \theta = a \sin \phi \cos \theta = b \sin \theta \cdot \cos \theta;$$

$$\therefore c - a \cos \phi = b \cos \theta = b(2m + \cos \phi);$$

$$\therefore \cos \phi = \frac{c - 2bm}{a + b}.$$

And

$$\cos \theta = 2m + \cos \phi = \frac{c + 2am}{a + b}.$$

But

$$a \sin \phi = b \sin \theta;$$

$$\therefore a^2 - a^2 \cos^2 \phi = b^2 - b^2 \cos^2 \theta;$$

$$\therefore a^2(a+b)^2 - a^2(c-2bm)^2 = b^2(a+b)^2 - b^2(c+2am)^2;$$

$$\therefore (a^2 - b^2)(a+b)^2 = c^2(a^2 - b^2) - 4abcm(a+b);$$

$$\therefore 4abcm = (a-b)\{c^2 - (a+b)^2\}.$$

EXAMPLES. XXV. c. PAGE 334.

1. By putting $x = y \cos \theta$, the given equation becomes

$$\cos^3 \theta - \frac{3}{y^2} \cos \theta - \frac{1}{y^3} = 0.$$

But

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3}{y^2} = \frac{3}{4}; \text{ whence } y = 2.$$

$$\text{Also } \frac{\cos 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}; \text{ whence } \cos 3\theta = \frac{1}{2};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 60^\circ;$$

$$\therefore \theta = 20^\circ, 100^\circ, \text{ or } 140^\circ.$$

But $x = y \cos \theta = 2 \cos \theta$; and therefore the roots are

$$2 \cos 20^\circ, -2 \cos 40^\circ, -2 \cos 80^\circ.$$

2. By putting $x=y \sin \theta$, the given equation becomes

$$\sin^3 \theta - \frac{3}{y^2} \sin \theta + \frac{1}{y^3} = 0.$$

But

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$$

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also $\frac{\sin 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}$; whence $\sin 3\theta = \frac{1}{2}$.

Hence as in Art. 328, the roots of the equation are

$$2 \sin 10^\circ, \quad 2 \sin 50^\circ, \quad -2 \sin 70^\circ.$$

3. Put $x=y \cos \theta$, then $\cos^3 \theta - \frac{3}{y^2} \cos \theta + \frac{\sqrt{3}}{y^3} = 0$.

But $\cos^3 \theta - \frac{3}{4} \cos \theta + \frac{\cos 3\theta}{4} = 0$;

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also $\frac{\cos 3\theta}{4} = \frac{\sqrt{3}}{y^3} = \frac{\sqrt{3}}{8}$; whence $\cos 3\theta = \frac{\sqrt{3}}{2}$;

$$\therefore 3\theta = n \cdot 360^\circ \pm 30^\circ; \quad \therefore \theta = 10^\circ, 110^\circ, \text{ or } 130^\circ.$$

But $x=y \cos \theta = 2 \cos \theta$, and therefore the roots are

$$2 \cos 10^\circ, \quad -2 \cos 50^\circ, \quad -2 \cos 70^\circ.$$

4. Put $x=y \sin \theta$; then $\sin^3 \theta - \frac{3}{4y^2} \sin \theta + \frac{\sqrt{2}}{8y^3} = 0$.

But $\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0$;

$$\therefore 4y^2 = 4; \text{ whence } y = 1.$$

Also since $\frac{\sin 3\theta}{4} = \frac{\sqrt{2}}{8y^3} = \frac{\sqrt{2}}{8}$; whence $\sin 3\theta = \frac{1}{\sqrt{2}}$;

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 45^\circ;$$

$$\therefore \theta = 15^\circ, 45^\circ, 135^\circ, 165^\circ, 255^\circ, \dots$$

$$\therefore \sin \theta = \sin 15^\circ, \sin 45^\circ, \sin 255^\circ.$$

But $x=y \sin \theta = \sin \theta$, and therefore the roots are

$$\sin 15^\circ, \quad \sin 45^\circ, \quad -\sin 75^\circ.$$

5. Put $x=y \sin \theta$; then $\sin^3 \theta - \frac{3}{4a^2y^2} \sin \theta + \frac{\sin 3\theta}{4a^3y^3} = 0$.

But $\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0$;

$$\therefore 4a^2y^2 = 4; \text{ whence } y = \frac{1}{a}.$$

Also

$$\frac{\sin 3\theta}{4} = \frac{\sin 3A}{4a^3 y^3} = \frac{\sin 3A}{4};$$

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 3A;$$

$$\therefore \theta = A, \quad 60^\circ - A, \quad 120^\circ + A, \quad 180^\circ - A, \quad 240^\circ + A, \dots$$

$$\therefore \sin \theta = \sin A, \quad \sin (60^\circ - A), \quad \sin (240^\circ + A).$$

But $x = y \sin \theta = \frac{1}{a} \sin \theta$, and therefore the roots are

$$\frac{1}{a} \sin A, \quad \frac{1}{a} \sin (60^\circ - A), \quad -\frac{1}{a} \sin (60^\circ + A).$$

6. Put $x = y \cos \theta$; then $\cos^3 \theta - \frac{3a^2}{y^2} \cos \theta - \frac{2a^3 \cos 3A}{y^3} = 0$.

But

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3a^2}{y^2} = \frac{3}{4}; \text{ whence } y = 2a.$$

Also

$$\frac{\cos 3\theta}{4} = \frac{2a^3 \cos 3A}{y^3} = \frac{\cos 3A}{4};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 3A;$$

$$\therefore \theta \equiv d, 120^\circ \pm d,$$

But $x = y \cos \theta = 2a \cos \theta$, and therefore the roots are

$$2a \cos A, \quad 2a \cos(120^\circ \pm A).$$

7. (1) From the theory of quadratic equations, we have

$$\sin \alpha + \sin \beta = -\frac{b}{a} \quad \dots \dots \dots \quad (1)$$

By supposition,

$$\sin \alpha + 2 \sin \beta = 1,$$

$$\therefore \sin \beta = 1 + \frac{b}{a}.$$

But

$$\therefore (a+b)^2 + b(a+b) + ac = 0.$$

(2) Substituting from the equation $c \sin \alpha = a \sin \beta$ in (1), we have

$$a \sin \beta + c \sin \beta = -\frac{bc}{a};$$

$$\therefore a(a+c) \sin \beta = -bc.$$

But

$$\sin \alpha \sin \beta = \frac{c}{a};$$

$$\therefore a^2 \sin^2 \beta = c^2; \text{ whence } a + c = \pm b.$$

8. We have $\tan \alpha + \tan \beta = \frac{b}{a}.$

Also, by hypothesis, $a \tan \alpha + b \tan \beta = 2b;$

whence $(b - a) \tan \beta = b.$

But $a \tan^2 \beta - b \tan \beta + c = 0;$

$$\therefore ab^2 - b^2(b - a) + c(b - a)^2 = 0.$$

9. We have $\tan \alpha + \tan \beta + \tan \gamma = 0,$

and $\tan \alpha \tan \beta + \tan \alpha \tan \gamma + \tan \beta \tan \gamma = \frac{2a - x}{a};$

$$\therefore \tan \alpha \tan \beta - \tan^2 \gamma = \frac{2a - x}{a}.$$

Now $\tan^2 \alpha + \tan^2 \beta = (\tan \alpha + \tan \beta)^2 - 2 \tan \alpha \tan \beta$

$$= \tan^2 \gamma - 2 \tan^2 \gamma - \frac{2(2a - x)}{a};$$

$$\therefore a(\tan^2 \alpha + \tan^2 \beta) = -a \tan^2 \gamma - 4a + 2x.$$

But, by hypothesis, $a(\tan^2 \alpha + \tan^2 \beta) = 2x - 5a;$

$$\therefore 2x - 5a = -a \tan^2 \gamma - 4a + 2x;$$

$$\therefore \tan^2 \gamma = 1.$$

But $a \tan^3 \gamma + (2a - x) \tan \gamma + y = 0;$

$$\therefore a \tan \gamma + (2a - x) \tan \gamma + y = 0;$$

$$\therefore (3a - x) \tan \gamma + y = 0;$$

$$\therefore 3a - x = \pm y.$$

10. We have $\cos \alpha \cos \beta + \cos \alpha \cos \gamma + \cos \beta \cos \gamma = b.$

Also, by supposition, $\cos \alpha \cos \beta + \cos \alpha \cos \gamma = 2b;$

$$\therefore \cos \beta \cos \gamma = -b.$$

Again, $\cos \alpha \cos \beta \cos \gamma = -c;$

$$\therefore b \cos \alpha = c.$$

Now $\cos^2 \alpha + a \cos^2 \alpha + b \cos \alpha + c = 0;$

$$\therefore c^3 + abc^2 + b^3c + b^3c = 0.$$

11. As on page 329 we may shew that

$$\cos a, \quad \cos\left(\frac{2\pi}{3} + a\right), \quad \cos\left(\frac{2\pi}{3} - a\right)$$

are the roots of the cubic

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\alpha}{4} = 0.$$

$$\text{Write } \cos \theta = \frac{1}{\sec \theta}; \text{ then } \frac{\cos 3\theta}{4} \sec^3 \theta + \frac{3}{4} \sec^2 \theta - 1 = 0;$$

$$\therefore \sec a + \sec\left(\frac{2\pi}{3} + a\right) + \sec\left(\frac{2\pi}{3} - a\right) = -\frac{3}{4} \div \frac{\cos 3a}{4}.$$

12. The values of $\sin \theta$ found from the equation $\sin 3\theta = \sin 3\alpha$ are

$$\sin \alpha, \quad \sin \left(\frac{2\pi}{3} + \alpha \right), \quad \sin \left(\frac{4\pi}{3} + \alpha \right);$$

that is, these three quantities are the roots of the cubic equation

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\alpha}{4} = 0 \quad \dots \dots \dots \quad (1).$$

$\therefore S_1 = \text{Sum of the roots} = 0;$

and $S_2 = \text{Sum of products of the roots taken two at a time} = -\frac{3}{4}$;

$$\therefore \sin^2 a + \sin^2 \left(\frac{2\pi}{3} + a\right) + \sin^2 \left(\frac{4\pi}{3} + a\right) = S_1^2 - 2S_2 \\ = 0 - 2 \left(-\frac{3}{4}\right) = \frac{3}{2}.$$

13. In equation (1) of Example 12, put $\sin \theta = \frac{1}{\operatorname{cosec} \theta}$; then

$$\frac{\sin 3a}{4} \operatorname{cosec}^3 \theta - \frac{3}{4} \operatorname{cosec}^2 \theta + 1 = 0.$$

$$\therefore \cosec \alpha + \cosec \left(\frac{2\pi}{3} + \alpha \right) + \cosec \left(\frac{4\pi}{3} + \alpha \right) = \frac{3}{4} + \frac{\sin 3\alpha}{4}.$$

14. On p. 330 we have shewn that $\sin^2 \frac{\pi}{5}$ and $\sin^2 \frac{2\pi}{5}$ are the roots of the quadratic equation $16x^2 - 20x + 5 = 0$.

Put $x = \frac{1}{y}$; then $5y^2 - 20y + 16 = 0$, is an equation whose roots are

$$\cosec^2 \frac{\pi}{5} \text{ and } \cosec^2 \frac{2\pi}{5};$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{5} + \operatorname{cosec}^2 \frac{2\pi}{5} = \frac{20}{5} = 4.$$

15. If

$$\cos 3\theta = \cos 2\theta,$$

then

$$4 \cos^3 \theta - 2 \cos^2 \theta - 3 \cos \theta + 1 = 0,$$

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$$(\cos \theta - 1)(4 \cos^2 \theta + 2 \cos \theta - 1) = 0.$$

But the roots of $\cos 3\theta = \cos 2\theta$ considered as a cubic in $\cos \theta$ are

$$1, \quad \cos \frac{2\pi}{5}, \quad \cos \frac{4\pi}{5}.$$

[Art. 331.]

$\therefore \cos \frac{2\pi}{5}$ and $\cos \frac{4\pi}{5}$ are the roots of $4\cos^2\theta + 2\cos\theta - 1 = 0$;

$$\therefore \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{2}{4} = -\frac{1}{2}.$$

and

$$\cos \frac{2\pi}{\tilde{\alpha}} \cos \frac{4\pi}{\tilde{\alpha}} = -\frac{1}{4}.$$

16. (1) Let $7\theta = n\pi$, where n is any odd integer;

then

$$4\theta = n\pi - 3\theta, \text{ and } \cos 4\theta = -\cos 3\theta.$$

$$\therefore 8 \cos^4 \theta - 8 \cos^2 \theta + 1 = -4 \cos^3 \theta + 3 \cos \theta;$$

The roots of the equation $\cos 4\theta = -\cos 3\theta$ considered as a biquadratic in $\cos \theta$ are $\cos \frac{\pi}{7}$, $\cos \frac{3\pi}{7}$, $\cos \frac{5\pi}{7}$, $\cos \frac{7\pi}{7}$, the last of which corresponds to the factor $\cos \theta + 1$ in equation (1). Hence the equation whose roots are $\cos \frac{\pi}{7}$, $\cos \frac{3\pi}{7}$, $\cos \frac{5\pi}{7}$ is

$$8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1 = 0.$$

(2) Let y denote any one of the quantities

$$\sin^2 \frac{\pi}{14}, \sin^2 \frac{3\pi}{14}, \sin^2 \frac{5\pi}{14}.$$

then $2y = 1 - x$, where x denotes one of the quantities

$$\cos \frac{\pi}{7}, \cos \frac{3\pi}{7}, \cos \frac{5\pi}{7}.$$

But we have seen in the first part of this question that these quantities are the roots of the cubic

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Substituting $x=1-2y$, we have

$$8(1-2y)^3 - 4(1-2y)^2 - 4(1-2y) + 1 = 0,$$

or

17. Let y denote one of the quantities $\sin^2 \frac{\pi}{7}$, $\sin^2 \frac{2\pi}{7}$, $\sin^2 \frac{3\pi}{7}$, then $2y = 1 - x$; where x denotes one of the quantities

$$\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}.$$

But in Art. 331, we have shewn that these quantities are the roots of the cubic

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Substituting $x = 1 - 2y$, we have

$$8(1 - 2y)^3 + 4(1 - 2y)^2 - 4(1 - 2y) - 1 = 0,$$

or

$$64y^3 - 112y^2 + 56y - 7 = 0,$$

the roots of which are $\sin^2 \frac{\pi}{7}$, $\sin^2 \frac{2\pi}{7}$, $\sin^2 \frac{3\pi}{7}$.

$$\therefore \sin^4 \frac{\pi}{7} + \sin^4 \frac{2\pi}{7} + \sin^4 \frac{3\pi}{7} = \left(\frac{112}{64}\right)^2 - 2\left(\frac{56}{64}\right) = \frac{21}{16}.$$

Put $y = \frac{1}{z}$; then $7z^3 - 56z^2 + 112z - 64 = 0$ is an equation whose roots are

$$\operatorname{cosec}^2 \frac{\pi}{7}, \operatorname{cosec}^2 \frac{2\pi}{7}, \operatorname{cosec}^2 \frac{3\pi}{7}.$$

$$\therefore \operatorname{cosec}^4 \frac{\pi}{7} + \operatorname{cosec}^4 \frac{2\pi}{7} + \operatorname{cosec}^4 \frac{3\pi}{7} = \left(\frac{56}{7}\right)^2 - 2\left(\frac{112}{7}\right) = 32.$$

18. (1) As in Art. 331 the required equation is $\cos 5\theta = \cos 4\theta$.

Expressing $\cos 5\theta$ and $\cos 4\theta$ in terms of $\cos \theta$, we have

$$16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1. \quad [\text{See Art. 332.}]$$

By transposition and removal of the factor $\cos \theta - 1$ we obtain

$$16 \cos^4 \theta + 8 \cos^3 \theta - 12 \cos^2 \theta - 4 \cos \theta + 1 = 0,$$

which is the equation required.

(2) Put $\cos \theta = x$, then the above equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

Now $\cos \frac{\pi}{9} = -\cos \frac{8\pi}{9}$, $\cos \frac{3\pi}{9} = -\cos \frac{6\pi}{9}$, ...;

hence by writing $x = -y$, we see that

$$\cos \frac{\pi}{9}, \cos \frac{3\pi}{9}, \cos \frac{5\pi}{9}, \cos \frac{7\pi}{9}$$

are the roots of the equation $16y^4 - 8y^3 - 12y^2 + 4y + 1 = 0$.

The same result may be arrived at by putting $9\theta = n\pi$, where n is any odd integer. For $5\theta = n\pi - 4\theta$, so that $\cos 5\theta = -\cos 4\theta$. [See Example 16 (1).]

19. Let y denote one of the quantities

$$\cos^2 \frac{\pi}{9}, \cos^2 \frac{2\pi}{9}, \cos^2 \frac{3\pi}{9}, \cos^2 \frac{4\pi}{9} \dots \quad (1),$$

then $2y=1+x$, where x denotes one of the quantities

$$\cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{6\pi}{9}, \cos \frac{8\pi}{9}.$$

But these quantities are by the last example the roots of the equation

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0;$$

hence the given quantities are the roots of

$$16(2y-1)^4 + 8(2y-1)^3 - 12(2y-1)^2 - 4(2y-1) + 1 \equiv 0.$$

or

$$256y^4 - 448y^3 + 240y^2 - 40y + 1 = 0.$$

$$\begin{aligned}\therefore \cos^4 \frac{\pi}{9} + \cos^4 \frac{2\pi}{9} + \cos^4 \frac{3\pi}{9} + \cos^4 \frac{4\pi}{9} &= \left(\frac{448}{256}\right)^2 - 2\left(\frac{240}{256}\right) \\ &= \frac{49}{16} - \frac{30}{16} = \frac{19}{16}.\end{aligned}$$

If we put $y = \frac{1}{z}$, we obtain

$$z^4 - 40z^3 + 240z^2 - 448z + 256 = 0,$$

the roots of which are

$$\sec^2 \frac{\pi}{9}, \sec^2 \frac{2\pi}{9}, \sec^2 \frac{3\pi}{9}, \sec^2 \frac{4\pi}{9}.$$

$$\therefore \sec^4 \frac{\pi}{9} + \sec^4 \frac{2\pi}{9} + \sec^4 \frac{3\pi}{9} + \sec^4 \frac{4\pi}{9} = (40)^2 - 2 \times 240 = 1120.$$

20. As on p. 332, we may shew that the given quantities are the roots of the equation $\tan 5\theta = -\tan 4\theta$, considered as an equation in $\tan \theta$.

Put $\tan \theta = t$, then $\tan 5\theta = \tan(3\theta + 2\theta)$;

$$\therefore \tan 5\theta = \frac{\frac{3t-t^3}{1-3t^2} + \frac{2t}{1-t^2}}{1 - \frac{2t(3t-t^3)}{(1-3t^2)(1-t^2)}} = \frac{5t-10t^3+t^5}{1-10t^2+5t^4}.$$

Hence the equation $\tan 5\theta = -\tan 4\theta$ becomes

$$\frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} - \frac{4t^3 - 4t}{t^4 - 6t^2 + 1} = 0;$$

$$\therefore (t^8 - 16t^6 + 66t^4 - 40t^2 + 5) - (20t^6 - 60t^4 + 4t^2 - 4) \equiv 0$$

$$\therefore t^8 - 36t^6 + 126t^4 - 84t^2 + 9 = 0.$$

which is a biquadratic in t^2 having for roots

$$\tan^2 \frac{\pi}{9}, \tan^2 \frac{2\pi}{9}, \tan^2 \frac{3\pi}{9}, \tan^2 \frac{4\pi}{9}.$$

Put $t^2 = \frac{1}{x}$, then

$$9x^4 - 84x^3 + 126x^2 - 36 + 1 = 0,$$

an equation whose roots are

$$\cot^2 \frac{\pi}{9}, \cot^2 \frac{2\pi}{9}, \cot^2 \frac{3\pi}{9}, \cot^2 \frac{4\pi}{9}.$$

But $\cot^2 \frac{3\pi}{9} = \cot^2 \frac{\pi}{3} = \frac{1}{3}$.

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} + \frac{1}{3} = \frac{84}{9} = 9 \frac{1}{3}.$$

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} = 9.$$

21. (1) As in Ex. 1, p. 332, we can shew that $t^6 - 21t^4 + 35t^2 - 7 = 0$ is an equation whose roots are

$$\tan^2 \frac{\pi}{7}, \tan^2 \frac{2\pi}{7}, \tan^2 \frac{3\pi}{7};$$

\therefore writing $\frac{1}{c}$ for t , we obtain

$$7c^6 - 35c^4 + 21c^2 - 1 = 0,$$

which is a cubic in c^2 whose roots are

$$\cot^2 \frac{\pi}{7}, \cot^2 \frac{2\pi}{7}, \cot^2 \frac{3\pi}{7}.$$

$$\therefore \cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} = 5;$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8.$$

(2) Here $\cos \frac{\pi}{11} = -\cos \frac{10\pi}{11}$, $\cos \frac{3\pi}{11} = -\cos \frac{8\pi}{11}$, $\cos \frac{5\pi}{11} = -\cos \frac{6\pi}{11}$.

$$\begin{aligned} \therefore \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \cos \frac{4\pi}{11} \cos \frac{5\pi}{11} \\ = -\cos \frac{2\pi}{11} \cos \frac{4\pi}{11} \cos \frac{6\pi}{11} \cos \frac{8\pi}{11} \cos \frac{10\pi}{11}. \end{aligned}$$

Now $\cos \frac{2\pi}{11}, \cos \frac{4\pi}{11}, \dots, \cos \frac{10\pi}{11}$ are the roots of the equation

$$\cos 6\theta = \cos 5\theta.$$

[See Art. 331.]

Using the expressions for $\cos 6\theta$ and $\cos 5\theta$ given in Art. 332 and putting x for $\cos \theta$, we have

$$32x^6 - 48x^4 + 18x^2 - 1 = 16x^5 - 20x^3 + 5x,$$

or $32x^6 - 16x^5 - 48x^4 + 20x^3 + 18x^2 - 5x - 1 = 0.$

Removing the factor $x - 1$, which corresponds to $\cos \theta = 1$, we have

$$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = 0.$$

The product of the roots is $-\frac{1}{32}$, and therefore the value of the required expression is $\frac{1}{32}.$

MISCELLANEOUS EXAMPLES. I. PAGE 336.

1. By transposition, we have

$$a \left(\tan \alpha - \tan \frac{\alpha + \beta}{2} \right) = b \left(\tan \frac{\alpha + \beta}{2} - \tan \beta \right);$$

$$\therefore \frac{a \sin \left(\alpha - \frac{\alpha + \beta}{2} \right)}{\cos \alpha \cos \frac{\alpha + \beta}{2}} = \frac{b \sin \left(\frac{\alpha + \beta}{2} - \beta \right)}{\cos \beta \cos \frac{\alpha + \beta}{2}},$$

$$\therefore \frac{a}{\cos \alpha} = \frac{b}{\cos \beta}.$$

2. We have $\frac{a+b}{a} \sin^4 \alpha + \frac{a+b}{b} \cos^4 \alpha = 1;$

$$\therefore \left(1 + \frac{b}{a} \right) \sin^4 \alpha + \left(1 + \frac{a}{b} \right) \cos^4 \alpha = \sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha + \cos^4 \alpha;$$

$$\therefore \frac{b}{a} \sin^4 \alpha - 2 \sin^2 \alpha \cos^2 \alpha + \frac{a}{b} \cos^4 \alpha = 0;$$

$$\therefore \frac{\sin^4 \alpha}{a^2} - \frac{2 \sin^2 \alpha \cos^2 \alpha}{ab} + \frac{\cos^4 \alpha}{b^2} = 0;$$

$$\therefore \frac{\sin^2 \alpha}{a} = \frac{\cos^2 \alpha}{b} = \frac{1}{a+b}.$$

$$\therefore \frac{\sin^8 \alpha}{a^3} + \frac{\cos^8 \alpha}{b^3} = \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4} = \frac{1}{(a+b)^3}.$$

3. The left side = $\tan^{-1} \frac{2 \tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2}\right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2}\right)}$

$$= \tan^{-1} \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \left(\frac{\pi}{4} - \frac{\beta}{2}\right) \cos \left(\frac{\pi}{4} - \frac{\beta}{2}\right)}{\cos^2 \frac{\alpha}{2} \cos^2 \left(\frac{\pi}{4} - \frac{\beta}{2}\right) - \sin^2 \frac{\alpha}{2} \sin^2 \left(\frac{\pi}{4} - \frac{\beta}{2}\right)}$$

$$= \tan^{-1} \frac{\frac{1}{2} \sin \alpha \sin \left(\frac{\pi}{2} - \beta\right)}{\cos \left\{ \frac{\alpha}{2} + \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\} \cos \left\{ \frac{\alpha}{2} - \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\}}$$

[XI. a. Ex. 12.]

$$= \tan^{-1} \frac{\sin \alpha \cos \beta}{\cos \alpha + \cos \left(\frac{\pi}{2} - \beta\right)}.$$

4. By putting $n=1$, we have

$$\operatorname{cosec}^2 \alpha \sin^4 \theta + \sec^2 \alpha \cos^4 \theta = 1;$$

$$\therefore (1 + \cot^2 \alpha) \sin^4 \theta + (1 + \tan^2 \alpha) \cos^4 \theta = \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta;$$

$$\therefore \cot^2 \alpha \sin^4 \theta - 2 \sin^2 \theta \cos^2 \theta + \tan^2 \alpha \cos^4 \theta = 0;$$

$$\therefore \cot \alpha \sin^2 \theta - \tan \alpha \cos^2 \theta = 0;$$

$$\therefore \frac{\sin^2 \theta}{\sin^2 \alpha} = \frac{\cos^2 \theta}{\cos^2 \alpha} = 1;$$

$$\therefore \frac{\sin^{2n} \theta}{\sin^{2n} \alpha} = \frac{\cos^{2n} \theta}{\cos^{2n} \alpha} = 1;$$

$$\therefore \frac{\sin^{2n+2} \theta}{\sin^{2n} \alpha} + \frac{\cos^{2n+2} \theta}{\cos^{2n} \alpha} = \sin^2 \theta + \cos^2 \theta = 1.$$

5. We have $(a \cos \theta + b \sin \theta)^2 = c^2 = c^2 (\cos^2 \theta + \sin^2 \theta);$
 $\therefore (a^2 - c^2) \cos^2 \theta + 2ab \cos \theta \sin \theta + (b^2 - c^2) \sin^2 \theta = 0.$

Again, $a \cos^2 \theta + b \sin^2 \theta = c (\cos^2 \theta + \sin^2 \theta);$
 $\therefore (a - c) \cos^2 \theta + (b - c) \sin^2 \theta = 0.$

Hence by cross multiplication,

$$\frac{\cos^2 \theta}{2ab(b-c)} = \frac{\cos \theta \sin \theta}{(b^2 - c^2)(a-c) - (a^2 - c^2)(b-c)} = \frac{\sin^2 \theta}{-2ab(a-c)};$$

$$\therefore \frac{\cos^2 \theta}{2ab(b-c)} = \frac{\cos \theta \sin \theta}{(b-c)(a-c)(b-a)} = \frac{\sin^2 \theta}{-2ab(a-c)};$$

$$\therefore -4a^2b^2(b-c)(a-c) = (b-c)^2(a-c)^2(b-a)^2,$$

or

$$4a^2b^2 + (b-c)(a-c)(a-b)^2 = 0.$$

6. (i) $4\Sigma \sin(\beta - \gamma) \cos(\alpha - \beta) \cos(\alpha - \gamma)$
 $= 2\Sigma \sin(\beta - \gamma) \{\cos(2\alpha - \beta - \gamma) + \cos(\beta - \gamma)\}$
 $= 2\Sigma \sin(\beta - \gamma) \cos\{2\alpha - (\beta + \gamma)\} + 2\Sigma \sin(\beta - \gamma) \cos(\beta - \gamma)$
 $= \Sigma \{\sin(2\alpha - 2\gamma) + \sin(2\beta - 2\alpha)\} + \Sigma \sin 2(\beta - \gamma)$
 $= -2\Sigma \sin 2(\beta - \gamma) + \Sigma \sin 2(\beta - \gamma)$
 $= -\Sigma \sin 2(\beta - \gamma)$
 $= 4\Pi \sin(\beta - \gamma).$

[Ex. 2, page 304.]

(ii) $4\Sigma \sin \alpha \sin(\beta - \gamma) \cos(\beta + \gamma - \alpha)$
 $= 2\Sigma \sin \alpha \{\sin(2\beta - \alpha) + \sin(\alpha - 2\gamma)\}$
 $= 2\Sigma \sin \alpha \sin(2\beta - \alpha) + 2\Sigma \sin \alpha \sin(\alpha - 2\gamma)$
 $= \Sigma \{\cos 2(\alpha - \beta) - \cos 2\beta\} + \Sigma \{\cos 2\gamma - \cos 2(\alpha - \gamma)\}$
 $= \Sigma \{\cos 2(\alpha - \beta) - \cos 2(\gamma - \alpha)\} + \Sigma (\cos 2\gamma - \cos 2\beta)$
 $= 0.$

(iii) $4\Sigma \sin \alpha \sin(\beta - \gamma) \sin(\beta + \gamma - \alpha)$
 $= 2\Sigma \sin \alpha \{\cos(\alpha - 2\gamma) - \cos(2\beta - \alpha)\}$
 $= \Sigma \{\sin(2\alpha - 2\gamma) + \sin 2\gamma\} - \Sigma \{\sin 2\beta + \sin(2\alpha - 2\beta)\}$
 $= -\Sigma \sin 2(\gamma - \alpha) + \Sigma (\sin 2\gamma - \sin 2\beta) - \Sigma \sin 2(\alpha - \beta)$
 $= -2\Sigma \sin 2(\alpha - \beta)$
 $= 8\Pi \sin(\alpha - \beta).$

[Ex. 2, page 304.]

7. (1) Let x, y, z denote the lengths of PA, PB, PC respectively; and let the areas of the triangles PBC, PCA, PAB be denoted by $\delta_1, \delta_2, \delta_3$ respectively.

Then in the triangle PBC

$$\begin{aligned}\cot \omega &= \frac{\cos \omega}{\sin \omega} = \frac{a^2 + y^2 - z^2}{2ay \sin \omega} = \frac{a^2 + y^2 - z^2}{4\delta_1} : \\ \therefore \cot \omega &= \frac{a^2 + y^2 - z^2}{4\delta_1} = \frac{b^2 + z^2 - x^2}{4\delta_2} = \frac{c^2 + x^2 - y^2}{4\delta_3} \\ &= \frac{a^2 + b^2 + c^2}{4(\delta_1 + \delta_2 + \delta_3)} = \frac{a^2 + b^2 + c^2}{4\Delta} \\ &= \cot A + \cot B + \cot C.\end{aligned}$$

[XVIII. a., Ex. 34.]

(2) By squaring the result just obtained, we have

$$\begin{aligned}\cot^2 \omega &= \cot^2 A + \cot^2 B + \cot^2 C + 2\Sigma \cot B \cot C \\ &= \cot^2 A + \cot^2 B + \cot^2 C + 2,\end{aligned}$$

since

$$\Sigma \cot B \cot C = 1.$$

$$\therefore 1 + \cot^2 \omega = (1 + \cot^2 A) + (1 + \cot^2 B) + (1 + \cot^2 C);$$

$$\therefore \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

8. On page 195 of the *Elementary Trigonometry*, suppose that $ABFD$ is a vertical plane running N. and S., and that CG is drawn in a S.E. direction. Also suppose that $AB=a$, and $AC=169a$;

then

$$BC^2=(169a)^2-a^2=28560a^2;$$

$$\therefore CG^2=2BC^2=57120a^2;$$

$$\therefore CH^2=57121a^2,$$

$$\therefore CH=239a.$$

9. Let ABC be a horizontal section of the two walls, and let ϕ be the inclination of the wall AB to the meridian, so that $\gamma-\phi$ is the inclination of BC to the meridian.

The length of the shadow of the wall AB measured along the meridian is $a \cot \theta$; hence the breadth of the shadow (which is measured at right angles to AB) is $a \cot \theta \sin \phi$,

$$\therefore b=a \cot \theta \sin \phi.$$

Similarly,

$$c=a \cot \theta \sin (\gamma-\phi);$$

$$\therefore c=a \cot \theta (\sin \gamma \cos \phi - \cos \gamma \sin \phi)$$

$$=a \cot \theta \sin \gamma \cos \phi - b \cos \gamma;$$

$$\therefore c+b \cos \gamma=a \cot \theta \sin \gamma \cos \phi.$$

Also

$$b \sin \gamma=a \cot \theta \sin \gamma \sin \phi.$$

By squaring and adding, we have

$$c^2+2bc \cos \gamma+b^2=a^2 \cot^2 \theta \sin^2 \gamma.$$

MISCELLANEOUS EXAMPLES. K.

1. If x is the number of degrees in the vertical angle $x+12x+12x=180$, whence $x=\frac{180}{25}=7\cdot2$. Thus the angle is $7^\circ 12'$.

Again, the number of grades = $\frac{200}{180} \times \frac{180}{25}=8$.

2. We have $\frac{\alpha}{4}=\frac{\beta}{5}=\frac{\gamma}{6}=\frac{\alpha+\beta+\gamma}{15}=\frac{\pi}{15}$,

whence $\alpha=\frac{4\pi}{15}$, $\beta=\frac{\pi}{3}$, $\gamma=\frac{2\pi}{5}$.

3. The first side = $\left(\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}\right) \div \left(\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)$
 $= (\cos^2 A - \sin^2 A) \div (\cos^2 A + \sin^2 A)$
 $= \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$
 $= 1 - 2 \sin^2 A.$

4. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13};$
 $\therefore \tan A + \sec A = \frac{5}{12} + \frac{13}{12} = \frac{3}{2}.$

5. Let $AB=15$, $AC=30$, then since $\cos 60^\circ = \frac{1}{2}$, it is easy to see that CB is at right angles to BA .

$$\therefore CB = CA \sin 60^\circ = 30 \times \frac{\sqrt{3}}{2} = 25.98.$$

6. Let AB be the tower, BD the cliff, and C the point of observation; then if $BD=x$ ft., $CD=y$ ft., we have $50+x=y \tan \alpha$, $x=y \tan \beta$.

By division $1 + \frac{50}{x} = \frac{\tan \alpha}{\tan \beta} = \frac{1260}{1185} = \frac{84}{79}.$
 $\therefore \frac{50}{x} = \frac{5}{79}$, and $x=790$.

7. If x ft. be the length of the arc, $\frac{x}{30}$ = radian measure of 10° ; whence

$$x = 30 \times \frac{\pi}{180} \times 10 = 5.236.$$

8. Here $\tan \alpha = \frac{8}{15}$; $\therefore \sin \alpha = \frac{8}{17}$, $\cos \alpha = \frac{15}{17}$.

9. Here $4 \sin^2 \theta - (2+2\sqrt{3}) \sin \theta + \sqrt{3} = 0$;
whence $(2 \sin \theta - \sqrt{3})(2 \sin \theta - 1) = 0$.

10. The expression $= \frac{5 \tan \alpha + 7}{6 - 3 \tan \alpha} = \frac{5 \times 4 + 7 \times 15}{6 \times 15 - 3 \times 4}$
 $= \frac{5}{3} \times \frac{25}{26} = \frac{125}{78}.$

11. First side $= 1 + 2(\sin A + \cos A) + (\sin A + \cos A)^2$
 $= 1 + 2(\sin A + \cos A) + 1 + 2 \sin A \cos A$
 $= 2 + 2(\sin A + \cos A) + 2 \sin A \cos A$
 $= 2(1 + \sin A)(1 + \cos A).$

12. The expression $= \sec^2 A (2 - \sec^2 A) - \operatorname{cosec}^2 A (2 - \operatorname{cosec}^2 A)$
 $= (1 + \tan^2 A)(1 - \tan^2 A) - (1 + \cot^2 A)(1 - \cot^2 A)$
 $= (1 - \tan^4 A) - (1 - \cot^4 A)$
 $= (1 - \tan^4 A) - \left(1 - \frac{1}{\tan^4 A}\right)$
 $= (1 - \tan^4 A) \left(1 + \frac{1}{\tan^4 A}\right) = \frac{1 - \tan^8 A}{\tan^4 A}.$

$$13. \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sin \alpha - \cos \alpha}{\sqrt{(\sin \alpha - \cos \alpha)^2 + (\sin \alpha + \cos \alpha)^2}}$$

$$= \frac{\sin \alpha - \cos \alpha}{\sqrt{2} (\sin^2 \alpha + \cos^2 \alpha)} = \frac{\sin \alpha - \cos \alpha}{\sqrt{2}}.$$

$$14. \text{First side} = \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right) \div \left(\frac{1}{\sqrt{2}} + \frac{1}{2} \right) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \frac{\sec 45^\circ - \tan 45^\circ}{\cosec 45^\circ + \cot 45^\circ}.$$

15. Since 9 degrees = 10 grades

$$1' = \frac{1}{60} \times \frac{10}{9} \times 100 \times 100 \times 1'' = \frac{10^4}{54} \times 1''.$$

$$16. (2) \text{Second side} = \left(1 - \frac{\sin \theta}{\cos \theta} \right)^2 \div \left(1 - \frac{\cos \theta}{\sin \theta} \right)^2$$

$$= \left(\frac{\cos \theta - \sin \theta}{\cos \theta} \right)^2 \div \left(\frac{\sin \theta - \cos \theta}{\sin \theta} \right)^2$$

$$= \frac{\sec^2 \theta}{\cosec^2 \theta} = \frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}.$$

$$17. (1) \quad \sin \theta + \frac{1}{\sin \theta} = \frac{3}{\sqrt{2}};$$

$$\therefore \sin^2 \theta - \frac{3}{\sqrt{2}} \sin \theta + 1 = 0,$$

$$\sqrt{2} \sin^2 \theta - 3 \sin \theta + \sqrt{2} = 0,$$

$$(\sqrt{2} \sin \theta - 1)(\sin \theta - \sqrt{2}) = 0;$$

whence $\sin \theta = \frac{1}{\sqrt{2}}$, and $\theta = 45^\circ$, the other value being impossible.

$$(2) \quad \cos \theta + \frac{1}{\cos \theta} = \frac{5}{2};$$

$$\therefore 2 \cos^2 \theta - 5 \cos \theta + 2 = 0,$$

$$(2 \cos \theta - 1)(\cos \theta - 2) = 0;$$

whence $\cos \theta = \frac{1}{2}$, and $\theta = 60^\circ$, the other value being impossible.

$$18. \text{Radian measure of } 56^\circ = \frac{\pi}{180} \times 56 = \frac{22}{7} \times \frac{56}{180} = \frac{44}{45}.$$

The arc traversed in $36'' = \frac{36}{60} \times \frac{10}{60} \times 1760$ yards = 176 yards.

\therefore if d be the number of yards in the diameter,

$$\frac{176}{d} = \frac{22}{45}; \therefore d = 360.$$

19. See Art. 35.

$$20. \text{ (1) First side} = 1 \times \left(\frac{\sin^2 A}{\cos^2 A} - 1 \right) = \frac{\sin^2 A - \cos^2 A}{\cos^2 A} \\ = \sec^2 A (\sin^2 A - \cos^2 A) = \text{second side.}$$

$$\text{(2) Second side} = \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \beta} - \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha} \\ = \sin^2 \alpha - \sin^2 \beta \\ = \text{First side.} \quad [\text{Examples III. } b, 34.]$$

$$21. \text{ We have } \cos B = \frac{a}{c} = .405; \text{ and } a + c = 281.$$

$$\therefore c(1 + .405) = 281; \text{ whence } c = 200, a = 81.$$

$$\text{Also } b = \sqrt{c^2 - a^2} = \sqrt{33439} = 183 \text{ nearly.}$$

$$22. \text{ The radian measure} = \frac{\text{arc}}{\text{radius}} = \frac{495}{3 \times 1760} = \frac{3}{32} = .09375.$$

$$\text{With this as unit a right angle would be } \frac{\pi}{2} \div \frac{3}{32} = 16.7552.$$

$$23. \text{ (1) First side} = \sin \theta \cos \theta \left\{ \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \right\} = \cos^2 \theta + \sin^2 \theta = 1.$$

$$\text{(2) First side} = \frac{\cot \theta}{\sec \theta} \times \frac{\cot^2 \theta}{\operatorname{cosec} \theta} \times \frac{\cos \theta}{\sin^3 \theta} \\ = \cot^3 \theta \times \frac{\cos \theta}{\sin \theta} \times \frac{\cos \theta}{\sin \theta} = \cot^5 \theta.$$

24. If O be the point of observation, $\angle BOA = 45^\circ$, and the line drawn from B perpendicular to OA bisects it at a point A' . Then

$$OA : OB = 2OA' : OB = 2 \cos 45^\circ = \sqrt{2} : 1.$$

$$25. \text{ (1) First side} = (\sin^2 A + \operatorname{cosec}^2 A - 2) + (\cos^2 A + \sec^2 A - 2) \\ = (\sin^2 A + \cos^2 A) + (\operatorname{cosec}^2 A - 1) + (\sec^2 A - 1) - 2 \\ = \cot^2 A + \tan^2 A - 1.$$

$$\text{(2) First side} = 3 \cot^2 \theta - 10 \cot \theta + 3 = 3 \operatorname{cosec}^2 \theta - 10 \cot \theta.$$

$$27. \text{ The expression} = \frac{2 - \cot A}{2 + 3 \cot A} = \frac{\frac{2 - \frac{9}{2}}{27}}{2 + \frac{2}{2}} = -\frac{5}{31}.$$

28. $2 \cos \theta \cot \theta + 1 - \cot \theta - 2 \cos \theta = 0;$

$$\therefore 2 \cos \theta (\cot \theta - 1) - (\cot \theta - 1) = 0;$$

$$\therefore (2 \cos \theta - 1)(\cot \theta - 1) = 0;$$

$$\therefore \theta = 60^\circ, \text{ or } 45^\circ.$$

29. If x inches be the length of the arc, $\frac{x}{6} = \frac{\pi}{180} \times \frac{1217}{60}.$

In the second circle $\theta = \frac{\pi}{180} \times \frac{1217}{10} \times \frac{1}{8};$

$$\therefore \text{sexagesimal measure of } \theta = \frac{1217}{10 \times 8} = 15^\circ 12' 45''.$$

30. Take the figure of the Example on p. 41, and let

$$PT = x \text{ yards}, \quad RT = y \text{ yards}.$$

Then $x = y + 110$, since $\angle PQT = 45^\circ$.

Also

$$y = x \cot 60^\circ = \frac{x}{\sqrt{3}}.$$

$$\therefore x \left(\frac{\sqrt{3}-1}{\sqrt{3}} \right) = 110;$$

$$\therefore x = 55(3 + \sqrt{3}) = 260.26.$$

31. First side = $\frac{1 + \cos A}{1 - \cos A} - \frac{1 - \cos A}{1 + \cos A} - 4 \cot^2 A$
 $= \frac{(1 + \cos A)^2 - (1 - \cos A)^2}{(1 - \cos A)(1 + \cos A)} - 4 \cot^2 A$
 $= \frac{2 \cdot 2 \cos A}{\sin^2 A} - \frac{4 \cos^2 A}{\sin^2 A} = \frac{4 \cos A (1 - \cos A)}{1 - \cos^2 A}$
 $= \frac{4 \cos A}{1 + \cos A} = \frac{4}{1 + \sec A}.$

32. (1) $8(1 - \cos^2 \theta) - 2 \cos \theta = 5;$

$$8 \cos^2 \theta + 2 \cos \theta - 3 = 0;$$

$$(2 \cos \theta - 1)(4 \cos \theta + 3) = 0;$$

whence $\theta = 60^\circ$; or $\cos \theta = -\frac{3}{4}.$

(2) $5 \tan^2 x - (1 + \tan^2 x) = 11;$

$$4 \tan^2 x = 12;$$

$$\tan x = \pm \sqrt{3}.$$

From the first of these values $x = 60^\circ$.

33. Here $\theta = \frac{\text{arc}}{\text{radius}} = \frac{11}{5 \times 12};$

$$\therefore \text{the angle in degrees} = \frac{180}{\pi} \times \frac{11}{60} = \frac{180}{22} \times \frac{11 \times 7}{60} = 10\frac{1}{2}^\circ.$$

35. See Art. 16.

(2) Here $\cos \theta = \frac{2ab}{a^2 + b^2}.$

Now since $(a - b)^2$ is a positive quantity, $a^2 + b^2 > 2ab$; $\therefore \cos \theta < 1$, which is possible.

36. Let ACB be the hill, A being the summit and C a point halfway down. Draw AD , CE perpendicular to the horizontal line through the object O . Then $AD = 2CE$, and $BD = 2BE$.

Now $OB + BE = CE \cot \beta$, and $OB + 2BE = 2CE \cot \alpha$;

$$\text{therefore, by subtraction, } \frac{OB + 2BE}{CE} - \frac{OB + BE}{CE} = 2 \cot \alpha - \cot \beta;$$

$$\text{that is, } \frac{BE}{CE} = 2 \cot \alpha - \cot \beta; \text{ also } \frac{BE}{CE} = \cot \theta.$$

37. From the figure in Ex. 2, Art. 46, we have

$$\cos B = \frac{a}{c} = \frac{25\sqrt{2}}{50} = \frac{1}{\sqrt{2}}; \quad \therefore B = 45^\circ;$$

$$\therefore b = c \sin 45^\circ = 50 \times \frac{1}{\sqrt{2}} = 25\sqrt{2}.$$

If p be the perpendicular from C , $p = a \sin 45^\circ = 25$.

40. Since $\tan \theta = \pm 1$, the angles will be those coterminal with 45° , 135° , 225° , 315° .

41. Take the figure of Example II. on p. 43.

Let $AE = x$, $CE = y$; then

$$\frac{x}{y} = .965; \quad \frac{x+42}{y} = 1.6.$$

$$\therefore \frac{x+42}{x} = \frac{1600}{965} = 1 + \frac{635}{965};$$

whence $x = 42 \times \frac{965}{635} = 63$, approximately;

$$\therefore AB = 63 + 42 = 105.$$

42. We have $\tan 15^\circ = \frac{\sin 30^\circ}{1 + \cos 30^\circ} = \frac{1}{2} / \left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$.

Similarly

$$\tan 75^\circ = 2 + \sqrt{3}.$$

Again,

$$1 + \tan^2 \theta = 4 \tan \theta,$$

$$\tan^2 \theta - 4 \tan \theta + 1 = 0;$$

$$\therefore \tan \theta = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3};$$

$$\therefore \theta = 75^\circ \text{ or } 15^\circ.$$

43. The first side

$$\begin{aligned} &= 1 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta \\ &\quad + \operatorname{cosec} \theta \tan \theta + \cot \theta + \sec \theta \cot \theta + 1 \\ &= 2 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta + \sec \theta + \cot \theta + \operatorname{cosec} \theta \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta \cos \theta} \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{2(\sin^2 \theta + \cos^2 \theta)}{\sin \theta \cos \theta} \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta + \tan \theta + \cot \theta). \end{aligned}$$

44. See figure and notation of Art. 25.

45. We have $2(1 - \sin^2 \theta) = 1 + \sin \theta;$

$$\therefore (1 + \sin \theta)(1 - 2 \sin \theta) = 0;$$

$$\therefore \sin \theta = -1, \text{ or } \sin \theta = \frac{1}{2};$$

whence

$$\theta = 30^\circ, 150^\circ, 270^\circ.$$

46 $\sin(270^\circ + A) = -\sin(90^\circ + A) = -\cos A.$

But

$$\cos A = \pm \sqrt{1 - \sin^2 A} = \pm \cdot 8;$$

$$\therefore \sin(270^\circ + A) = \pm \cdot 8.$$

47. See Art. 113.

48. The expression $= \frac{2 \sin 2A \sin A}{2 \sin A \cos 2A} = \tan 2A.$ See Art. 89.

49. $\tan A = \sqrt{\sec^2 A - 1} = \pm \sqrt{\frac{4}{3} - 1} = \pm \frac{1}{\sqrt{3}}.$

The angle is coterminal with 150° or 210° , and the tangents of these angles are equal but opposite in sign.

50. Take the figure of Example on p. 41, and let

$$PT=x \text{ yards}, \quad \angle PQT=45^\circ, \quad \angle PRT=60^\circ.$$

Also

$$QR=1760, \text{ and } QT=PT;$$

$$\therefore \frac{x}{x-1760} = \tan 69^\circ = \sqrt{3};$$

$$\therefore x(\sqrt{3}-1) = 1760\sqrt{3};$$

$$\therefore x = 880(3+\sqrt{3}) = 4164.16.$$

52. (1) First side $= \sin^2 \alpha (\sin^2 \alpha + 2 \cos^2 \alpha)$
 $= (1 - \cos^2 \alpha)(1 + \cos^2 \alpha) = 1 - \cos^4 \alpha.$

(2) First side $= \sec 2 \left(\frac{\pi}{4} - \theta \right)$ [Art. 124]
 $= \operatorname{cosec} 2\theta.$

(3) $\cos 10^\circ + \sin 40^\circ = \cos 10^\circ + \cos 50^\circ = 2 \cos 30^\circ \cos 20^\circ$
 $= \sqrt{3} \sin 70^\circ.$

53. The expression $= \frac{a \tan \theta - b}{a \tan \theta + b} = \frac{a^2 - b^2}{a^2 + b^2}.$

54. Multiply all through by $\cos 18^\circ$, then we have to prove that

$$4 \cos^2 18^\circ - 3 = 2 \sin 18^\circ.$$

$$\begin{aligned} \text{First side} &= 4 \left\{ 1 - \left(\frac{\sqrt{5}-1}{4} \right)^2 \right\} - 3 = \frac{10+2\sqrt{5}}{4} - 3 \\ &= \frac{\sqrt{5}-1}{2} = 2 \sin 18^\circ. \end{aligned}$$

56. Here $\theta = \frac{\text{arc}}{\text{radius}} = \frac{20 \times 10}{60 \times 60} \div \frac{1}{2} = \frac{1}{9};$

$$\therefore D = \frac{1}{9} \times 180 \times \frac{7}{22} = \frac{70}{11} = 6 \frac{4}{11}^\circ.$$

57. The expression $= \frac{2-3 \cot \alpha}{4-9 \tan \alpha}$, where $\tan \alpha = \frac{12}{5}$,

$$= \frac{2-3 \times \frac{5}{12}}{4-9 \times \frac{12}{5}} = \frac{3}{4} \div \left(-\frac{88}{5} \right)$$

$$= -\frac{15}{352}.$$

58. (2) First side = $\frac{2 \sin \frac{\pi}{4} \sin \theta}{2 \cos \frac{2\pi}{3} \sin \theta} + \sqrt{2}$

$$= \frac{1}{\sqrt{2}} \left| \left(-\frac{1}{2} \right) + \sqrt{2} \right| = 0.$$

59. (1) The expression = $\frac{\cos A \cos C + (-\cos C)(-\cos A)}{\cos A \sin C - \sin C(-\cos A)}$
 $= \frac{2 \cos A \cos C}{2 \cos A \sin C} = \cot C.$

(2) The expression = $\frac{\sin A \cos A + \text{two similar terms}}{\sin A \sin B \sin C}$
 $= \frac{1}{2} \left(\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C} \right)$
 $= 2.$ [Art. 135, Ex. 1.]

60. Let ABC be the horizontal equilateral triangle, and let PQ be the flagstaff. Then since each side subtends an angle of 60° at P , the top of the flagstaff, the triangles PCB , PBA , PAC are equilateral.

Let x be a side of $\triangle ABC$;

then $AQ = \frac{x}{2 \sec 30^\circ} = \frac{x}{\sqrt{3}}.$

Then from $\triangle PAQ$,

$$PA^2 = AQ^2 + QP^2,$$

$$x^2 = \frac{x^2}{3} + 10000;$$

whence

$$x^2 = 15000, \text{ or } x = 50\sqrt{6}.$$

61. The first expression = $\sin \theta + \cos \theta + \sin^2 \theta + \cos^2 \theta$
 $= \sin \theta + \cos \theta + 1.$

Similarly, the second expression = $\sin \theta + \cos \theta - 1$;

\therefore the product = $(\sin \theta + \cos \theta)^2 - 1 = 2 \sin \theta \cos \theta = \sin 2\theta.$

62. Second side = $\sin^2 \frac{\theta + \phi}{2} + \cos^2 \frac{\theta - \phi}{2} - 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2}$
 $= \frac{1}{2} \{1 - \cos(\theta + \phi) + 1 + \cos(\theta - \phi)\} - \sin \theta - \sin \phi.$
 $= 1 - \sin \theta - \sin \phi + \frac{1}{2} \{\cos(\theta - \phi) - \cos(\theta + \phi)\}$
 $= 1 - \sin \theta - \sin \phi + \sin \theta \sin \phi$
 $= (1 - \sin \theta)(1 - \sin \phi).$

63. The expression $= \frac{2 \cos \alpha \sin \theta}{2 \sin \beta \sin \theta} = \frac{\cos \alpha}{\sin \beta}$, which is independent of θ .

$$\begin{aligned} 64. \text{ First side} &= \sin \frac{B+C}{2} \cos \frac{B-C}{2} + \dots + \dots \\ &= \frac{1}{2} \{(\sin B + \sin C) + \dots + \dots\} \\ &= \sin A + \sin B + \sin C. \end{aligned}$$

$$\begin{aligned} 65. \text{ We have } \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} &= \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}; \\ 2 \sin^2 \frac{\theta}{2} &= \left(2 \cos^2 \frac{\theta}{2} - 1\right)^2; \\ \therefore 4 \cos^4 \frac{\theta}{2} - 2 \cos \frac{\theta}{2} - 1 &= 0; \\ \therefore \cos^2 \frac{\theta}{2} &= \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}; \\ \therefore \cos^2 \frac{\theta}{2} &= \cos 36^\circ, \text{ the other value being impossible.} \end{aligned}$$

66. Draw ZW perpendicular to XY and let $ZW=x$. Then $\triangle XZY$ is right-angled at Z .

$$\text{And } XZ = XY \cos 30^\circ = 100\sqrt{3} \text{ yds.}$$

$$\text{Again, from } \triangle WXZ, WZ = XZ \sin 30^\circ = 50\sqrt{3} = 86.6 \text{ yds.}$$

67. We have $32\pi \times 1000 = \frac{5585}{2} \times 3 \times 12$; whence $\pi = 3.141$, approximately.

$$\begin{aligned} 68. \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma &= \frac{1 - \cos 2\alpha}{2} + \dots + \dots \\ &= \frac{3}{2} - \frac{1}{2} (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) \\ &= \frac{3}{2} - \frac{1}{2} \{2 \cos(\alpha + \beta) \cos(\alpha - \beta) + 1 - 2 \sin^2 \gamma\} \\ &= 1 - \{\sin \gamma \cos(\alpha - \beta) - \sin^2 \gamma\} \\ &= 1 - \sin \gamma \{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} \\ &= 1 - 2 \sin \alpha \sin \beta \sin \gamma. \end{aligned}$$

69. (1) First side = $\left(\frac{\sin A}{\cos A} + \frac{\sin 2A}{\cos 2A} \right) (2 \cos 2A \cos A)$
 $= 2(\sin 2A \cos A + \cos 2A \sin A) = 2 \sin 3A.$

(2) Multiply all through by 32; then

$$\begin{aligned}\text{Second side} &= 2 + \cos 2A - 2 \cos 4A - \cos 6A \\&= 2(1 - \cos 4A) + 2 \sin 4A \sin 2A \\&= 4 \sin^2 2A + 4 \sin^2 2A \cos 2A \\&= 4 \sin^2 2A (1 + \cos 2A) \\&= 16 \sin^2 A \cos^2 A \cdot 2 \cos^2 A \\&= 32 \sin^2 A \cos^4 A.\end{aligned}$$

70. The expression = $\frac{2 \cos 13a \sin 10a}{2 \sin 10a \cos 6a} = \frac{\cos 13a}{\cos 6a}$
 $= \frac{\cos 13a}{\cos (\pi - 13a)} = -1.$

71. We have $\cot(A+B)=1$. Therefore

$$\begin{aligned}\cot A \cot B - 1 &= \cot A + \cot B; \\ \therefore 2 \cot A \cot B &= 1 + \cot A + \cot B + \cot A \cot B \\ &= (1 + \cot A)(1 + \cot B); \\ \therefore \frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} &= \frac{1}{2}.\end{aligned}$$

72. For the first part see XI. d. Ex. 15.

Then $\tan \theta = \cot \theta - 2 \cot 2\theta,$

$$2 \tan 2\theta = 2 \cot 2\theta - 4 \cot 4\theta,$$

$$4 \tan 4\theta = 4 \cot 4\theta - 8 \cot 8\theta;$$

\therefore by addition

$$\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta = \cot \theta - 8 \cot 8\theta.$$

73. The expression = $1 - \frac{\sin^3 \theta}{\sin \theta + \cos \theta} - \frac{\cos^3 \theta}{\sin \theta + \cos \theta}$
 $= 1 - \frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta}$
 $= 1 - (\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta)$
 $= \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta.$

74. We have $x=3 \sin A - (3 \sin A - 4 \sin^3 A) = 4 \sin^3 A$,

$$y=4 \cos^3 A - 3 \cos A + 3 \cos A = 4 \cos^3 A;$$

$$\therefore \left(\frac{x}{4}\right)^{\frac{3}{2}} + \left(\frac{y}{4}\right)^{\frac{3}{2}} = \sin^2 A + \cos^2 A = 1;$$

$$\therefore x^{\frac{3}{2}} + y^{\frac{3}{2}} = 4^{\frac{3}{2}}.$$

75. Let PQ, RS be the flagstaffs of lengths x, y feet respectively. Then $QABS$ is a straight line, such that $AB=30$ ft., $\angle PAQ=60^\circ$, $\angle RAS=30^\circ$, $\angle PBQ=45^\circ$, $\angle RBS=60^\circ$. Let $AQ=a$, $BS=b$. Then since

$$\angle PBQ=45^\circ, BQ=QP=x;$$

$$\therefore a=x-30.$$

From $\triangle ARS$, we have $AS=RS \cot 30^\circ=y\sqrt{3}$;

$$\therefore b=y\sqrt{3}-30.$$

From $\triangle APQ$, $PQ=AQ \tan 60^\circ$;

$$\therefore x=(x-30)\sqrt{3};$$

whence

$$x=15(3+\sqrt{3}).$$

From $\triangle BRS$, $RS=BS \tan 60^\circ$;

$$\therefore y=\sqrt{3}(y\sqrt{3}-30);$$

whence

$$y=15\sqrt{3}.$$

Again,

$$\begin{aligned} QS &= a+b+30 \\ &= x+y\sqrt{3}-30 \\ &= 45+15\sqrt{3}+15-30 \\ &= 60+15\sqrt{3}. \end{aligned}$$

76. (1) First side

$$\begin{aligned} &= \frac{1}{2}(1+\cos 2A) + \frac{1}{2}(1+\cos 2B) - 2 \cos A \cos B \cos(A+B) \\ &= 1 + \cos(A+B) \cos(A-B) - 2 \cos A \cos B \cos(A+B) \\ &= 1 + \cos(A+B) \{ \sin A \sin B - \cos A \cos B \} \\ &= 1 - \cos^2(A+B) = \sin^2(A+B). \end{aligned}$$

(2) First side

$$\begin{aligned} &= 2(\sin 5A - \sin A) - (\sin 3A + \sin A) \\ &= 4 \cos 3A \sin 2A - 2 \sin 2A \cos A \\ &= 2 \sin 2A (2 \cos 3A - \cos A) \\ &= 4 \sin A \cos A (8 \cos^3 A - 7 \cos A) \\ &= 4 \sin A \cos^2 A \{ 8(1 - \sin^2 A) - 7 \} \\ &= 4 \sin A \cos^2 A (1 - 8 \sin^2 A). \end{aligned}$$

77. If r is the radius of the circle, and AB one side of the square, we have $2\pi r = 3$, and

$$\begin{aligned} AB &= 2r \sin 45^\circ = \frac{3\sqrt{2}}{2\pi} \\ &= \frac{3}{2} \times 1.4142 \times 3183 \\ &= 6752 \text{ feet} \\ &= 8.10 \text{ inches.} \end{aligned}$$

78. Here $AB = 2r \sin 54^\circ$, $BC = 2r \sin 30^\circ$, $CD = 2r \sin 18^\circ$. And it remains to prove that

$$\sin 54^\circ = \sin 30^\circ + \sin 18^\circ.$$

[See Examples XI. e. 9.]

79. First side $= (2 + \sqrt{3}) + (2 - \sqrt{3}) - 1 - 2 = 1$.

80. We have, by addition, $\cot \theta = 2(m+n)$.

Also, by subtraction, $\cos \theta = 2(m-n)$.

$$\begin{aligned} \therefore 4(m^2 - n^2) &= \frac{\cos^2 \theta}{\sin \theta}; \\ \therefore 16(m^2 - n^2)^2 &= \frac{\cos^4 \theta}{\sin^2 \theta} = \cot^2 \theta \times \cos^2 \theta \\ &= \cot^2 \theta (1 - \sin^2 \theta) \\ &= 16mn. \end{aligned}$$

81. (1) First side $= \frac{1}{2} [\sin(2\beta + \alpha) + \sin \alpha - \sin(2\gamma + \alpha) - \sin \alpha]$

$$= \frac{1}{2} [2 \cos(\alpha + \beta + \gamma) \sin(\beta - \gamma)].$$

$$\begin{aligned} (2) \quad \text{First side} &= \left(\frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} \right) \left(\frac{1}{\cos A} + \frac{1}{\cos 3A} \right) \\ &= \frac{\sin A}{\cos 2A \cos A} \cdot \frac{\cos 3A + \cos A}{\cos A \cos 3A} \\ &= \frac{2 \sin A \cos 2A \cos A}{\cos 2A \cos^2 A \cos 3A} = 2 \sin A \sec A \sec 3A. \end{aligned}$$

82. $2 \cos 6^\circ \cos 66^\circ = \cos 72^\circ + \cos 60^\circ = \sin 18^\circ + \frac{1}{2}$

$$= \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4}.$$

$$2 \cos 42^\circ \cos 78^\circ = \cos 120^\circ + \cos 36^\circ$$

$$= -\frac{1}{2} + \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1}{4}.$$

$$\therefore 4 \cos 6^\circ \cos 66^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{4}.$$

83. Put $2A = 45^\circ$.

84. The distance required is evidently equal to $10 \tan 22\frac{1}{2}^\circ$

$$= \frac{10}{\sqrt{2}+1} = 10(\sqrt{2}-1) = 4.14 \text{ miles.}$$

85. (1) Separate each term into the difference of two cosines.

$$\begin{aligned} (2) \text{ Second side} &= \frac{\sin \theta + 2 \sin \theta \cos \theta}{\cos \theta + 2 \cos^2 \theta} \\ &= \frac{\sin \theta (1 + 2 \cos \theta)}{\cos \theta (1 + 2 \cos \theta)} = \tan \theta. \end{aligned}$$

$$86. \text{ First side} = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1$$

$$= 2 \cos \frac{\gamma}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\gamma}{2} \right\} - 1$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2} \right\} - 1$$

$$= 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 1.$$

$$87. \text{ Put } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k, \text{ then}$$

$$\text{first side} = \frac{k^2 (\sin^2 B - \sin^2 C)}{k \sin A} \cos A + \dots + \dots$$

$$= k \left\{ \frac{\sin(B+C) \sin(B-C)}{\sin A} \cos A + \dots + \dots \right\}$$

$$= k \{ \sin(B-C) \cos A + \dots + \dots \}$$

$$= -k \{ \sin(B-C) \cos(B+C) + \dots + \dots \}$$

$$= -\frac{k}{2} \{ (\sin 2B - \sin 2C) + \dots + \dots \}$$

$$= 0.$$

$$\begin{aligned}
 88. \quad & \text{First side} = a \cos 2\theta + b \sin 2\theta \\
 &= a(1 - 2 \sin^2 \theta) + 2b \sin \theta \cos \theta \\
 &= a + 2 \sin \theta(b \cos \theta - a \sin \theta) \\
 &= a, \text{ since } b \cos \theta = a \sin \theta.
 \end{aligned}$$

89. Let $\log_a b = x$, so that $a^x = b$,

$$\begin{aligned}
 \log_b c &= y, \dots \dots \dots b^y = c, \\
 \log_c a &= z, \dots \dots \dots c^z = a.
 \end{aligned}$$

Then we have $a = c^z = b^{yz} = a^{xyz}$;

$$\therefore xyz = 1, \text{ or } \log_a b \log_b c \log_c a = 1.$$

We have $\log 8 = \log 2^3 = 3 \log 2$; whence $\log 2 = .30103$;

$$\begin{aligned}
 \log 2.4 &= \log \left(\frac{3 \times 8}{10} \right) = \log 3 + \log 8 - 1 \\
 &= 1.47712 + .90309 = .38021.
 \end{aligned}$$

$$\begin{aligned}
 \log 5400 &= 2 + \log 2 + 3 \log 3 \\
 &= 2.30103 + 1.43136 = 3.73239.
 \end{aligned}$$

$$\begin{aligned}
 L \tan 30^\circ &= 10 + \log \frac{1}{\sqrt{3}} = 10 - \frac{1}{2} \log 3 \\
 &= 9.76144.
 \end{aligned}$$

90. $\cot(A+B) = \cot(90^\circ - C) = \tan C = \frac{1}{\cot C}$;

$$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1}{\cot C};$$

whence by multiplying up and rearranging we obtain the required result.

For the second part, put $A = 15^\circ$, $B = 30^\circ$, $C = 45^\circ$.

$$\begin{aligned}
 91. \quad & \text{First side} = (1 + \sin 2A)^2 + \cos^2 2A + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A)^2 + (1 - \sin^2 2A) + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A) \{1 + \sin 2A + 1 - \sin 2A + 2 \cos 2A\} \\
 &= 2(1 + \cos 2A)(1 + \sin 2A) = 4 \cos^2 A (1 + \sin 2A).
 \end{aligned}$$

92. See Art. 150.

93. We have to prove that $\sin 9^\circ \sin 81^\circ = \sin 12^\circ \sin 48^\circ$.

$$\begin{aligned}
 \text{First side} &= \frac{1}{2} (\cos 72^\circ - \cos 90^\circ) \\
 &= \frac{1}{2} \sin 18^\circ = \frac{\sqrt{5}-1}{8}.
 \end{aligned}$$

$$\begin{aligned}\text{Second side} &= \frac{1}{2}(\cos 36^\circ - \cos 60^\circ) \\ &= \frac{1}{2} \left(\frac{\sqrt{5}+1}{4} - \frac{1}{2} \right) = \frac{\sqrt{5}-1}{8}.\end{aligned}$$

94. See Art. 136, Ex. 2.

95. $L \sin \theta > L \sin 27^\circ 45'$ by $\frac{1742}{2400} \times 60''$;

$$\text{whence } \theta = 27^\circ 45' 44''.$$

96. By a well-known algebraical formula,

$$x^3 + y^3 + z^3 = 3xyz,$$

when

$$x + y + z = 0;$$

therefore we have

$$\cos^3 A + \cos^3 B + \cos^3 C = 3 \cos A \cos B \cos C.$$

Substituting $\frac{1}{4}(\cos 3A + 3 \cos A)$ for $\cos^3 A$, and similar results for $\cos^3 B$, $\cos^3 C$,

we have

$$\frac{1}{4}(\cos 3A + \cos 3B + \cos 3C) + \frac{3}{4}(\cos A + \cos B + \cos C) = 3 \cos A \cos B \cos C,$$

whence the required result follows at once, since

$$\cos A + \cos B + \cos C = 0.$$

97. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{49}{625}} = \pm \frac{24}{25}$; but since A lies between 270° and 360° , we must reject the negative value; thus $\cos A = \frac{24}{25}$.

$$\begin{aligned}\text{Hence } \sin 2A &= 2 \sin A \cos A = 2 \left(-\frac{7}{25} \right) \left(\frac{24}{25} \right) \\ &= -\frac{336}{625}.\end{aligned}$$

$$\text{Also } \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A} = \operatorname{cosec} A - \cot A$$

$$= -\frac{25}{7} + \frac{24}{7} = -\frac{1}{7}.$$

98. We have

$$\frac{2 \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \left(\frac{\sin \theta + \cos \theta}{\sin \theta} \right)^2;$$

$$8 \cos^3 \frac{\theta}{2} \sin \frac{\theta}{2} = 1 + \sin 2\theta,$$

$$4 \cos^2 \frac{\theta}{2} \sin \theta = 1 + \sin 2\theta,$$

$$2 \sin \theta (1 + \cos \theta) = 1 + \sin 2\theta;$$

$$\therefore 2 \sin \theta = 1,$$

or

$$\theta = 30^\circ.$$

Again, by putting $\theta = 30^\circ$, we have

$$2 \cot 15^\circ = (1 + \sqrt{3})^2 = 4 + 2\sqrt{3}.$$

$$\therefore \cot 15^\circ = 2 + \sqrt{3}, \text{ and } \tan 15^\circ = 2 - \sqrt{3}.$$

99. We have $\log 360 = 2 \log 2 + 2 \log 3 + 1$.

$$\begin{aligned}\therefore 2 \log 3 &= \log 360 - 2 \log 2 - 1 \\ &= 1.5563025 - .6020600 \\ &= .9542425;\end{aligned}$$

$$\therefore \log 3 = .4771213.$$

$$\text{Now } \log .04 = \log 4 - 2 = 2 \log 2 - 2 = 2.60206.$$

$$\begin{aligned}\log 24 &= 3 \log 2 + \log 3 = .90309 + .4771213 \\ &= 1.3802113.\end{aligned}$$

$$\begin{aligned}\log \dot{6} &= \log \frac{2}{3} = \log 2 - \log 3 \\ &= .30103 - .4771213 = .8239087.\end{aligned}$$

Again let $\log_2 30 = x$, so that $2^x = 30$.

$$\therefore x \log 2 = \log 30 = 1 + \log 3;$$

$$\therefore x = \frac{1 + \log 3}{\log 2} = \frac{1.4771213}{.30103} = 4.90689.$$

100. This follows at once from Art. 134, Ex. 5.

$$\begin{aligned}
 101. \cot(\alpha+\beta) &= \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta} = \frac{x(x+x^{-1}+1)-1}{(x+x^{-1}+1)^{\frac{1}{2}}(1+x)} = \frac{x^2+x}{(1+x)(x+x^{-1}+1)^{\frac{1}{2}}} \\
 &= \frac{x}{(x+x^{-1}+1)^{\frac{1}{2}}} = \frac{1}{x^{-1}(x+x^{-1}+1)^{\frac{1}{2}}} = \frac{1}{(x^{-1}+x^{-3}+x^{-2})^{\frac{1}{2}}} \\
 &= \cot\gamma.
 \end{aligned}$$

Therefore $\alpha+\beta=\gamma$.

102. Let A and a be given, and let B be the right angle; then $c=a\cot A$, $b=a\operatorname{cosec} A$, or $b=\sqrt{a^2+c^2}$. Also $C=90^\circ-A$.

If $A=31^\circ 53' 26.8''$, $a=28$, we have

$$c=28\cot A.$$

$$\log c = \log 28 + \log \cot 31^\circ 53' 26.8'',$$

$$\log 28 = 1.4471580$$

$$\log \cot 31^\circ 53' = .2061805$$

$$\text{decrease for } 26.8'' \quad \underline{1258}$$

$$\therefore \log c = 1.6532127; \therefore c=45.$$

Again

$$b^2=a^2+c^2=2025+784=2809;$$

$$\therefore b=53; \text{ also } C=90^\circ-A=58^\circ 6' 33.2''.$$

104. The greatest angle, C , is opposite to $\sqrt{x^2+xy+y^2}$.

$$\therefore \cos C = \frac{x^2+y^2-(x^2+xy+y^2)}{2xy} = -\frac{1}{2};$$

$$\therefore C=120^\circ.$$

$$\begin{aligned}
 105. \cos 3A + \sin 3A &= 4\cos^3 A - 3\cos A + 3\sin A - 4\sin^3 A \\
 &= 4(\cos^3 A - \sin^3 A) - 3(\cos A - \sin A),
 \end{aligned}$$

which is evidently divisible by $\cos A - \sin A$.

See solution to Examples XII. c, 27.

$$\begin{aligned}
 106. \text{We have } \cot \frac{C}{2} &= \tan \frac{A+B}{2} = \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6} \cdot \frac{20}{37}} = \frac{5}{2}.
 \end{aligned}$$

$$\therefore \tan C = \frac{2 \tan \frac{C}{2}}{1 - \tan^2 \frac{C}{2}} = \frac{2 \times \frac{2}{5}}{1 - \frac{4}{25}} = \frac{20}{21}.$$

For the second part, it will be sufficient to prove that

$$\sin A + \sin C = 2 \sin B.$$

$$\text{Now } \sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{60}{61}, \text{ on reduction.}$$

Similarly $\sin C = \frac{20}{29}$, and $\sin B = \frac{1480}{1769}$, whence the result follows.

107. The first factor easily reduces to $2 \cot 2\theta$, and the second to $\frac{2}{\cos 2\theta}$; whence the product becomes $4 \operatorname{cosec} 2\theta$.

$$108. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{128}{603}},$$

$$\log \tan \frac{A}{2} = \frac{1}{2} \{ \log 128 - \log 603 \} = 1.6634465;$$

whence $\frac{A}{2} = 24^\circ 44' 16''$, and $A = 49^\circ 28' 32''$.

$$109. \quad \tan(A-B) = \frac{\tan A - \frac{n \sin A \cos A}{1-n \sin^2 A}}{1 + \tan A \left(\frac{n \sin A \cos A}{1-n \sin^2 A} \right)} = \frac{\tan A \left(1 - \frac{n \cos^2 A}{1-n \sin^2 A} \right)}{1 + \frac{n \sin^2 A}{1-n \sin^2 A}}$$

$$= (1-n) \tan A.$$

$$110. \quad \log 200 = 2 + \log 2 = 3 - \log 5 = 2.30103.$$

$$\log .025 = 2 \log 5 - 3 = 2.39794,$$

$$\log \sqrt[3]{62.5} = \frac{1}{3} (\log 625 - 1) = \frac{1}{3} (4 \log 5 - 1)$$

$$= .598626.$$

$$L \sin 30^\circ = 10 + \log \left(\frac{1}{2} \right) = 10 + \log 5 - 1$$

$$= 9.69897.$$

$$L \cos 45^\circ = 10 + \log \left(\frac{1}{\sqrt{2}} \right) = 10 + \frac{1}{2} (\log 5 - 1)$$

$$= 9.849485.$$

$$111. \quad (1) \quad \frac{\cot(A-30^\circ)}{\tan(A+30^\circ)} = \frac{\cos(A-30^\circ) \cos(A+30^\circ)}{\sin(A-30^\circ) \sin(A+30^\circ)}$$

$$= \frac{\cos 2A + \cos 60^\circ}{\cos 60^\circ - \cos 2A} = \frac{2 \cos 2A + 1}{1 - 2 \cos 2A}$$

$$= \frac{2 + \sec 2A}{\sec 2A - 2}.$$

$$(2) \text{ Second side} = 2 \left[\frac{1 + \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\beta}{2} + \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 - \cos 2\beta}{2} \right] \\ = \frac{1}{2} \{2 + 2 \cos 2\alpha \cos 2\beta\} = 1 + \cos 2\alpha \cos 2\beta.$$

112. From a diagram it is easily seen that CD is equal to AC , and that from the right-angled triangle ABC ,

$$AC = BC \cos 30^\circ = 132 \frac{\sqrt{3}}{2}$$

$$= 66 \times \frac{19}{11} = 114 \text{ yards.}$$

Also the perp. from A on $BC = AC \sin 30^\circ = 57$ yards.

$$113. \text{ Since } \frac{a}{a+b+c} = \frac{\sin A}{\sin A + \sin B + \sin C} \\ = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}, \quad [\text{XII. } d. \text{ Ex. 3.}] \\ \therefore \frac{a \cot \frac{A}{2} + b \cot \frac{B}{2} - c \cot \frac{C}{2}}{a+b+c} = \frac{1}{2} \left\{ \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \dots - \dots \right\} \\ = \frac{1}{2} \left\{ \frac{\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \right\}.$$

$$\text{Now } \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 1 + \cos B - 1 - \cos C \right\} \\ = \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{C-B}{2} \right\} \\ = \cos \frac{A}{2} \left\{ \cos \frac{A}{2} + \sin \frac{C-B}{2} \right\} \\ = \cos \frac{A}{2} \left\{ \sin \frac{B+C}{2} + \sin \frac{C-B}{2} \right\} \\ = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

Whence the required result easily follows.

$$114. \text{ Here } \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{40 \times 42}{75 \times 77}}$$

$$= \frac{2}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{10}},$$

$$\log \cos \frac{A}{2} = \frac{3}{2} \log 2 - \frac{1}{2} \log 10$$

$$= .4515450 - .5$$

$$= \bar{1}.9515450$$

$$\log \cos 26^\circ 34' = \bar{1}.9515389$$

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$$\therefore \frac{A}{2} \text{ is less than } 26^\circ 34' \text{ by } \frac{61}{632} \times 60'';$$

$$\text{that is, } \frac{A}{2} = 26^\circ 33' 54.2'',$$

or

$$A = 53^\circ 7' 48''.$$

$$115. \sin(A - 90^\circ) = -\sin(90^\circ - A) = -\cos A$$

$$= -\sqrt{1 - \sin^2 A} = -\sqrt{6.4}$$

$$= -(\pm 8);$$

but A is between 90° and 180° , therefore $\sin(A - 90^\circ)$ is positive; that is

$$\sin(A - 90^\circ) = 8.$$

$$\operatorname{cosec}(270^\circ - A) = \operatorname{cosec}(180^\circ + 90^\circ - A)$$

$$= -\operatorname{cosec}(90^\circ - A)$$

$$= -\left(\frac{1}{\pm 8}\right)$$

$$= \pm 1.25;$$

but between the given limits $\operatorname{cosec}(270^\circ - A)$ must be positive, that is, the required value is 1.25.

116. By Art. 168,

$$\log_b c = \frac{\log_a c}{\log_a b}, \quad \log_c d = \frac{\log_a d}{\log_a c};$$

$$\begin{aligned} \text{hence the expression on the right} &= \log_a b \times \frac{\log_a c}{\log_a b} \times \frac{\log_a d}{\log_a c} \\ &= \log_a d. \end{aligned}$$

$$\log_{10} 2 = 1 - \log_{10} 5 = .30103;$$

$$\log_{10} 8 = 3 \log_{10} 2 = .90309.$$

$$\log_8 10 = \frac{1}{\log_{10} 8} = 1.1073093.$$

$$\begin{aligned}\log_{10} (.032)^5 &= 5 \log_{10} \frac{32}{1000} = 25 \log 2 - 15 \\ &= 7.52575 - 15 \\ &= 8.52575.\end{aligned}$$

$$\begin{aligned}117. \text{ First side} &= \cos(360^\circ + 60^\circ + A) + \cos(60^\circ - A) \\ &= \cos(60^\circ + A) + \cos(60^\circ - A) \\ &= 2 \cos 60^\circ \cos A = \cos A.\end{aligned}$$

For the second part, put $A = 45^\circ$.

118. Write t for $\tan \frac{x}{2}$, then the equation may be written

$$\frac{1-t^2}{1+t^2} - \sin \alpha \cot \beta \frac{2t}{1+t^2} = \cos \alpha;$$

$$\therefore t^2(1+\cos \alpha) + 2t \sin \alpha \cot \beta - (1-\cos \alpha) = 0;$$

$$t^2 + 2 \cot \beta \frac{\sin \alpha}{1+\cos \alpha} \cdot t - \frac{1-\cos \alpha}{1+\cos \alpha} = 0,$$

$$t^2 + 2 \cot \beta \tan \frac{\alpha}{2} t - \tan^2 \frac{\alpha}{2} = 0,$$

$$\left(t + \tan \frac{\alpha}{2} \cot \frac{\beta}{2} \right) \left(t - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right) = 0, \quad [\text{XI. d. Ex. 15.}]$$

$$\therefore \tan \frac{x}{2} = -\tan \frac{\alpha}{2} \cot \frac{\beta}{2}, \text{ or } \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

119. We have $\frac{\sin(2A+B)}{\sin B} = \frac{m}{n};$

$$\therefore \frac{m-n}{m+n} = \frac{\sin(2A+B) - \sin B}{\sin(2A+B) + \sin B}$$

$$= \frac{2 \cos(A+B) \sin A}{2 \sin(A+B) \cos A}$$

$$= \cot(A+B) \tan A.$$

120. Let x feet be the height of the tower;
then $\angle ABD = 90^\circ - \angle ADB$

$$= 45^\circ = \angle ADB;$$

$$\therefore AD = AB = x;$$

$$\therefore AC = x - 17.$$

Now from $\triangle ABC$,

$$x^2 + (x - 17)^2 = 53^2;$$

$$\therefore x^2 - 17x - 1260 = 0,$$

$$(x - 45)(x + 28) = 0;$$

$$\therefore x = 45.$$

Again,

$$\tan ACB = \frac{45}{28};$$

but $\tan 31^\circ 48' = \frac{56}{90} = \frac{28}{45};$

$$\therefore \angle ACB = 90^\circ - 31^\circ 48' \\ = 58^\circ 12'.$$

122. $\log 6 = \frac{1}{2} \log 36 = .778151,$

$$\log 8 = \log 48 - \log 6 = 1.681241 - .778151 \\ = .90309;$$

$$\therefore \log 2 = \frac{1}{3} \log 8 = .30103;$$

$$\therefore \log 3 = \log 6 - \log 2 = .477121.$$

Now $\log 40 = 1 + 2 \log 2 = 1.60206.$

$$\begin{aligned} \log \sqrt{\frac{2}{15}} &= \frac{1}{2} (\log 4 - \log 30) \\ &= \frac{1}{2} (2 \log 2 - \log 3 - 1) \\ &= 1.562469, \text{ on substitution.} \end{aligned}$$

123. We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{9}{10} \cot 72^\circ.$

$$\begin{aligned} \log \tan \frac{B-C}{2} &= 2 \log 3 - 1 + \log \cot 72^\circ \\ &= 1.4660186 \end{aligned}$$

$$\log \tan 16^\circ 18' = \frac{1.4660078}{108} \quad \frac{108}{4687} \times 60'' = 1''.$$

$$\therefore \frac{B-C}{2} = 16^\circ 18' 1'', \quad \frac{B+C}{2} = 18^\circ;$$

$$\therefore B = 34^\circ 18' 1'', \quad C = 1^\circ 41' 59''.$$

$$124. \quad (1) \quad \cos A + \cos B \cos C = -\cos B + \cos C + \cos B \cos C \\ = \sin B \sin C;$$

$$\therefore \text{Second side} = a^2 \sin B \sin C = b \sin A \cdot c \sin A \\ = bc \sin^2 A.$$

$$(2) \quad \text{First side} = c(b \cos A + a \cos B) + 2ab \cos C \\ = c^2 + 2ab \cos C = a^2 + b^2.$$

$$125. \quad \tan \frac{\beta - \alpha}{2} = \frac{\tan \frac{\beta}{2} - \tan \frac{\alpha}{2}}{1 + \tan \frac{\beta}{2} \tan \frac{\alpha}{2}} = \frac{3 \tan \frac{\alpha}{2}}{1 + 4 \tan^2 \frac{\alpha}{2}}$$

$$= \frac{3 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + 4 \sin^2 \frac{\alpha}{2}} = \frac{3 \sin \alpha}{2 \cos^2 \frac{\alpha}{2} + 8 \sin^2 \frac{\alpha}{2}}$$

$$= \frac{3 \sin \alpha}{1 + \cos \alpha + 4(1 - \cos \alpha)} = \frac{3 \sin \alpha}{5 - 3 \cos \alpha}.$$

$$126. \quad \sin(36^\circ + A) - \sin(36^\circ - A) = 2 \cos 36^\circ \sin A = \frac{\sqrt{5} + 1}{2} \sin A;$$

$$\sin(72^\circ - A) - \sin(72^\circ + A) = 2 \cos 72^\circ \sin(-A) = -\frac{\sqrt{5} - 1}{2} \sin A.$$

By addition we obtain the required result.

$$127. \quad \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{-\frac{2}{3}}{\pm \sqrt{\frac{5}{9}}} = \pm \frac{2}{\sqrt{5}}.$$

The boundary line of θ is in the 3rd or 4th quadrant, hence the tangent is positive in one case and negative in the other.

$$128. \quad (1) \quad \text{First side} = \sin 2A + \sin \left(\frac{\pi}{2} - 2B\right) \\ = 2 \sin \left(\frac{\pi}{4} + A - B\right) \cos \left(\frac{\pi}{4} - A - B\right).$$

$$(2) \quad \text{First side} = 2 \sin \frac{\theta - \phi}{2} \cos \frac{\theta + \phi}{2} \cdot 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\ = 2 \sin(\theta - \phi) \cos^2 \frac{\theta + \phi}{2}.$$

129. Since $a : b : c = \sin A : \sin B : \sin C$, we have

$$(a+b+c)(a+b-c)=3ab,$$

or

$$(a+b)^2 - c^2 = 3ab,$$

that is, $\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}$; $\therefore \cos C = \frac{1}{2}$, and $C = 60^\circ$.

130. Let $a^x = b$, $a^y = d$, $c^p = d$, $c^q = b$,

then we have to prove $px = qy$.

Now

$$a^x = c^q, \text{ and } a^y = c^p;$$

$$\therefore a^{px} = c^{pq} = a^{qy}; \text{ that is } px = qy.$$

131.

$$\log 20.01 = 1.3012471$$

$$\log 20.00 = \frac{1.3010300}{2171}$$

$$\therefore \text{diff. for } .01 = \frac{3}{4} \times 2171 = 1628;$$

$$\therefore \log 20.0075 = 1.3011928.$$

132. Let AD be the median from B ; then

$$AB^2 + BC^2 = 2(AD^2 + BD^2);$$

that is,

$$49 + 81 = 2x^2 + 32;$$

whence

$$x = 7.$$

133. We have

$$\frac{1 + \sin A}{\cos A} = 2,$$

$$\therefore (1 + \sin A)^2 = 4(1 - \sin^2 A),$$

rejecting the factor $1 + \sin A$, which gives an inadmissible value, we have

$$1 + \sin A = 4(1 - \sin A);$$

whence

$$\sin A = \frac{3}{5}.$$

$$134. \text{ First side} = \frac{4(1 - \cos 2A) - (1 - \cos 4A)}{4(1 + \cos 2A) - (1 - \cos 4A)}$$

$$= \frac{8 \sin^2 A - 2 \sin^2 2A}{8 \cos^2 A - 2 \sin^2 2A}$$

$$= \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A (1 - \sin^2 A)} = \tan^4 A.$$

$$\begin{aligned}
 135. \text{ First side} &= \frac{3 \sin A - 4 \sin^3 A + 4 \cos^3 A - 3 \cos A}{3 \sin A - 4 \sin^3 A - (4 \cos^3 A - 3 \cos A)} \\
 &= \frac{3(\sin A - \cos A) - 4(\sin^3 A - \cos^3 A)}{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)} \\
 &= \frac{\sin A - \cos A}{\sin A + \cos A} \cdot \frac{3 - 4(\sin^2 A + \cos^2 A + \sin A \cos A)}{3 - 4(\sin^2 A + \cos^2 A - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{\tan A + 1} \cdot \frac{3 - 4(1 + \sin A \cos A)}{3 - 4(1 - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{1 + \tan A} \cdot \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} = \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} \tan(A - 45^\circ).
 \end{aligned}$$

$$136. \text{ Put } \frac{x}{\cos A} = \frac{y}{\cos B} = k; \text{ then}$$

$$\text{First side} = k \cos A \tan A + k \cos B \tan B$$

$$\begin{aligned}
 &= k(\sin A + \sin B) \\
 &= 2k \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\
 &= 2k \cos \frac{A+B}{2} \cos \frac{A-B}{2} \tan \frac{A+B}{2} \\
 &= k(\cos A + \cos B) \tan \frac{A+B}{2} \\
 &= (x+y) \tan \frac{A+B}{2}.
 \end{aligned}$$

$$137. \log 7 = \log 24.5 - \log 3.5 = .845098;$$

$$\log 5 = \log 35 - \log 7 = .69897;$$

$$\log 13 = \log 3.25 - \log .25$$

$$= \log 3.25 - (2 \log 5 - 2) = 1.113943.$$

$$138. \text{ Here } \tan A = \frac{a}{b} = \frac{384}{330} = \frac{128}{110}.$$

$$\log 128 = 7 \log 2 = 2.1072100$$

$$\log 110 = 2.0413927$$

$$\log \tan A = .0658173$$

$$\log \tan 49^\circ 19' = .0656886$$

$$\text{diff.} \quad 1287$$

$$\therefore \text{prop}^l \text{ increase} = \frac{1287}{2555} \times 60'' = 30'';$$

$$\therefore A = 49^\circ 19' 30''; B = 40^\circ 40' 30''.$$

$$\begin{aligned}
 139. \text{ First side} &= \frac{\frac{2 \sin \frac{\theta+\alpha}{2} \sin \frac{\theta-\alpha}{2}}{2 \cos \frac{\theta+\alpha}{2} \cos \frac{\theta-\alpha}{2}} = \frac{\cos \alpha - \cos \theta}{\cos \alpha + \cos \theta}}{=} \\
 &= \frac{\cos \alpha - \cos \alpha \cos \beta}{\cos \alpha + \cos \alpha \cos \beta} = \frac{1 - \cos \beta}{1 + \cos \beta} \\
 &= \tan^2 \frac{\beta}{2}.
 \end{aligned}$$

140. We have

$$\begin{aligned}
 \frac{\sin \theta}{\cos \theta - \sin \phi} - \frac{\sin \theta}{\cos \theta + \sin \phi} &= \frac{2 \sin \theta \sin \phi}{\cos^2 \theta - \sin^2 \phi} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{1 - \sin^2 \theta - (1 - \cos^2 \phi)} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{(\cos \phi + \sin \theta)(\cos \phi - \sin \theta)} \\
 &= \frac{\sin \phi}{\cos \phi - \sin \theta} - \frac{\sin \phi}{\cos \phi + \sin \theta}.
 \end{aligned}$$

$$141. \text{ We have } \frac{c^2(a+b)^2 s(s-b)}{ac} = \frac{b^2(a+c)^2 s(s-c)}{ab};$$

$$c(a^2 + b^2 + 2ab)(s-b) = b(a^2 + c^2 + 2ac)(s-c);$$

$$\text{that is, } a^2s(c-b) - bcs(c-b) + bc(c^2 - b^2) + 2abc(c-b) = 0,$$

$$\text{or } (c-b)\{a^2s - bc(s-c-b) + 2abc\} = 0,$$

$$\text{or } (c-b)\{a^2s - bc(a-s) + 2abc\} = 0,$$

$$\text{or } (c-b)\{a^2s + bcs + 3abc\} = 0;$$

therefore $b-c=0$, since the other factor evidently cannot be zero.

$$142. (1) \cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2}; \quad [\text{XI. d. Ex. 11}]$$

$$\begin{aligned}
 \tan A + \sec A &= \frac{1 + \sin A}{\cos A} = \frac{1 - \cos \left(\frac{\pi}{2} + A \right)}{\sin \left(\frac{\pi}{2} + A \right)} \\
 &= \tan \left(\frac{\pi}{4} + \frac{A}{2} \right);
 \end{aligned}$$

therefore by division the required result is obtained.

(2) Since $\sin 3A = 3 \sin A - 4 \sin^3 A$, we have

$$\begin{aligned} \text{First side} &= \frac{1}{4} \{ (3 \sin A - \sin 3A) + \dots + \dots \} \\ &= \frac{3}{4} \{ \sin A + \sin (120^\circ + A) + \sin (240^\circ + A) \} \\ &\quad - \frac{1}{4} \{ \sin 3A + \sin (360^\circ + 3A) + \sin (720^\circ + 3A) \} \\ &= -\frac{3}{4} \sin 3A. \quad [\text{See solution of XII. c. 25.}] \end{aligned}$$

143. Let

$$x = \sqrt[5]{18 \times 0015},$$

then

$$x = \sqrt[5]{027} = (3)^{\frac{3}{5}},$$

$$\frac{3}{5} \log 3 = \frac{3}{5} (1.4771213)$$

$$\begin{aligned} \log 1.6862728 &= \text{diff. for } 00001 = 31; \\ \log 1.6862697 &= \frac{31}{31} \quad \therefore \text{prop}^l \text{ increase} = \frac{31}{90} \times 00001 \\ &= 000003; \end{aligned}$$

$$\therefore x = 485593.$$

$$144. \quad \sin B = \frac{b \sin A}{a} = \frac{394}{573} \cos 22^\circ 4'.$$

$$\log 394 = 2.5954962$$

$$\log \cos 22^\circ 4' = \frac{1.9669614}{2.5624576}$$

$$\log 573 = 2.7581546$$

$$\log \sin B = 1.8043030$$

$$\log \sin 39^\circ 35' = \frac{1.8042757}{273}$$

$$\text{diff.} \quad 273$$

$$\text{diff. for } 60'' = 1527;$$

$$\therefore \text{prop}^l \text{ increase} = \frac{273}{1527} \times 60'' = 10.7'';$$

$$\therefore B = 39^\circ 35' 11''; \text{ and } C = 28^\circ 20' 49''.$$

$$145. \quad \text{First side} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

146.

$$\log 119 = \log 7 + \log 17 = 2.0755469,$$

$$\log \frac{17}{7} = \log 17 - \log 7 = .3853509,$$

$$\log \frac{289}{343} = \log 17^2 - \log 7^3$$

$$= 2 \log 17 - 3 \log 7 = 1.9256038.$$

147. We have

$$\cos \theta = \frac{\sin A}{\sin B + \sin C}$$

$$= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} = \frac{\cos \frac{B+C}{2}}{\cos \frac{B-C}{2}};$$

$$\therefore \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\cos \frac{B-C}{2} - \cos \frac{B+C}{2}}{\cos \frac{B-C}{2} + \cos \frac{B+C}{2}}$$

$$= \frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

148. Let r be the radius of the circle, x, y the side of circumscribing equilateral triangle and hexagon respectively.

Then from the figure of Art. 215,

$$x = 2r \tan 60^\circ = 2r \sqrt{3}; \quad y = 2r \tan 30 = \frac{2r}{\sqrt{3}};$$

whence

$$xy = 4r^2 = (2r)^2.$$

149. From the equation $a^2 = b^2 + c^2 - 2bc \cos A$, we have on substitution and reduction

$$c^2 - 150\sqrt{2} \cdot c + 10000 = 0;$$

$$\therefore (c - 100\sqrt{2})(c - 50\sqrt{2}) = 0;$$

$$\therefore c = 100\sqrt{2}, \text{ or } 50\sqrt{2}.$$

Again

$$\sin B = \frac{b \sin A}{a} = \frac{150}{50\sqrt{10}} = \frac{3}{\sqrt{10}},$$

$$\log \sin B = \log 3 - \frac{1}{2} \log 10$$

$$= 1.9771213,$$

$$\log \sin 71^\circ 33' = \frac{1.9770832}{381}$$

$$\text{diff. for } 60'' = 421;$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{381}{421} \times 60'' = 54'';$$

$$\therefore B = 71^\circ 33' 54'', \text{ or } 180^\circ - (71^\circ 33' 54'').$$

150. The series may be written

$$(\operatorname{cosec} x - \operatorname{cosec} 3x) + (\operatorname{cosec} 3x - \operatorname{cosec} 3^2 x) + \dots + (\operatorname{cosec} 3^{n-1} x - \operatorname{cosec} 3^n x),$$

which reduces to

$$\operatorname{cosec} x - \operatorname{cosec} 3^n x.$$

153. First side

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{3 \cos \frac{3\theta}{2}}{\sin \frac{3\theta}{2}}$$

$$= \frac{\sin \frac{3\theta}{2} \cos \frac{\theta}{2} - 3 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \left(\frac{3\theta}{2} - \frac{\theta}{2} \right) - 2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \theta - (\sin 2\theta - \sin \theta)}{\frac{1}{2} (\cos \theta - \cos 2\theta)}$$

$$= \frac{4 \sin \theta - 2 \sin 2\theta}{1 + \cos \theta - 2 \cos^2 \theta} = \frac{4 \sin \theta (1 - \cos \theta)}{(1 + 2 \cos \theta)(1 - \cos \theta)}$$

$$= \frac{4 \sin \theta}{1 + 2 \cos \theta}.$$

154. We have, by Art. 135, Ex. 2,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

$$\therefore \frac{3}{4} + \frac{5}{12} + \tan C = \frac{3}{4} \cdot \frac{5}{12} \cdot \tan C;$$

whence

$$\tan C = -\frac{56}{33}.$$

Also

$$\cos A = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5},$$

$$\cos B = \frac{1}{\sqrt{1 + \tan^2 B}} = \frac{1}{\sqrt{1 + \frac{25}{144}}} = \frac{12}{13}.$$

$$\cos C = \frac{1}{\sqrt{1 + \tan^2 C}} = \frac{1}{\sqrt{1 + \frac{3136}{1089}}} = -\frac{33}{65},$$

the negative sign of the radical being taken in the third case since C is an obtuse angle.

Again

$$\tan C = -\frac{56}{33}; \quad \therefore \quad \tan(180^\circ - C) = \frac{56}{33},$$

$$\log 56 = 1.7481880$$

$$\log 33 = 1.5185139$$

$$\begin{array}{rcl} \log \tan (180^\circ - C) = .2296741 & & \therefore \text{prop'l increase} = \frac{1114}{2888} \times 60'' \\ \log \tan 59^\circ 29' = .2295627 & & = 23''; \\ \text{diff.} & 1114 & \end{array}$$

1114

= 23'';

$$\therefore 180^\circ - C = 59^\circ 29' 23''$$

that is,

$$C = 120^\circ 30' 37''.$$

155. We have

$$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2 - b^2}{a^2 + b^2}$$

$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B};$$

$$\therefore \frac{\sin(A-B)}{\sin C} = \frac{\sin(A-B) \sin(A+B)}{\sin^2 A + \sin^2 B};$$

; either

or

$$\sin^2 C = \sin^2 A + \sin^2 B, \dots \quad (2).$$

If (1) is true, $A \equiv B$; if (2) is true, we have $c^2 \equiv a^2 + b^2$.

$$\begin{aligned}
 156. \quad \text{The first side} &= \frac{2\Delta}{s} \cdot \frac{abc}{\Delta} \left\{ \frac{s(s-a)}{bc} + \dots + \dots \right\} \\
 &= 2 \{a(s-a) + b(s-b) + c(s-c)\} \\
 &= (a+b+c)^2 - 2a^2 - 2b^2 - 2c^2 \\
 &= 2bc + 2ca + 2ab - a^2 - b^2 - c^2.
 \end{aligned}$$

157. We have

$$\begin{aligned}
 \frac{\cos C}{\sin B \cos A} - \frac{\cos B}{\sin C \cos A} &= \frac{\cos(A+C)}{\sin C \cos A} - \frac{\cos(A+B)}{\sin B \cos A} \\
 &= \cot C - \tan A - (\cot B - \tan A) \\
 &= \cot C - \cot B.
 \end{aligned}$$

$$\begin{aligned}
 158. \quad \log 3 &= \log 18 - \log 6; \quad \log 2 = \log 6 - \log 3; \\
 \log 11 &= \log 44 - \log 4 = \log 44 - 2 \log 2.
 \end{aligned}$$

159. (1) We have

$$\begin{aligned}
 &\tan(60^\circ + A) \tan(60^\circ - A) \\
 &= \frac{2 \sin(60^\circ + A) \sin(60^\circ - A)}{2 \cos(60^\circ + A) \cos(60^\circ - A)} = \frac{\cos 2A - \cos 120^\circ}{\cos 2A + \cos 120^\circ} = \frac{2 \cos 2A + 1}{2 \cos 2A - 1}; \\
 \therefore \text{second side} &= 2 \sin A \cos 2A + \sin A \\
 &= (\sin 3A - \sin A) + \sin A = \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= 2 \sin(A-B) \cos(A+B) \frac{\sin(A+B)}{\cos(A+B)} \\
 &= 2 \sin(A+B) \sin(A-B) \\
 &= 2 \sin^2 A - \sin^2 B. \quad [\text{Art. 114.}]
 \end{aligned}$$

$$\begin{aligned}
 160. \quad \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{113 \times 101}{296 \times 82}} \\
 \log 296 &= 2.4712917 \quad \log 113 = 2.0530784 \\
 \log 82 &= \underline{1.9138139} \quad \log 101 = \underline{2.0043214} \\
 &\quad \underline{4.3851056} \quad \underline{4.0573998} \\
 &\quad \underline{4.3851056} \\
 &\quad 2 \underline{1.6722942} \\
 \log \tan \frac{C}{2} &= \underline{1.8361471} \\
 \log \tan 34^\circ 26' &= \underline{1.8360513} \\
 &\quad \text{diff.} \quad 958 \\
 \text{prop'l. increase} &= \frac{958}{2708} \times 60'' = 21''; \\
 \therefore C &= 68^\circ 52' 42''.
 \end{aligned}$$

161. Let a be a side of the octagon, r the radius of the circle, then $8a=2\pi r$; and we have

$$\frac{\text{area of circle}}{\text{area of octagon}} = \frac{\pi r^2}{2a^2 \cot \frac{\pi}{8}} = \frac{8\pi r^2}{\pi^2 r^2} \tan \frac{\pi}{8}$$

$$= \frac{8(\sqrt{2}-1)}{\pi} = \frac{414 \times 8}{3.1416} = \frac{4140}{3927} = \frac{1380}{1309}.$$

162. We have $2b=a+c$, or $a=2b-c$;

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b-c)^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}.$$

163. We have $a(\sin \theta \cos \alpha + \cos \theta \sin \alpha) = b(\sin \theta \cos \beta + \cos \theta \sin \beta)$;

$$\therefore \sin \theta(a \cos \alpha - b \cos \beta) = \cos \theta(b \sin \beta - a \sin \alpha);$$

that is,

$$\cot \theta = \frac{a \cos \alpha - b \cos \beta}{b \sin \beta - a \sin \alpha}.$$

164. Let a, b, A be the given parts; then $R = \frac{a}{2 \sin A}$, which is the same for each triangle.

165. Let NS be a horizontal line pointing North and South. Then if K be the position of the kite, and KD is the vertical from K , we have S, B, D, A, N in a straight line, and $BA=c$. Also $KA = \frac{c \sin \beta}{\sin BKA} = \frac{c \sin \beta}{\sin(\alpha+\beta)}$.

And $KD = KA \sin \alpha = \frac{c \sin \alpha \sin \beta}{\sin(\alpha+\beta)}$.

$$166. \cos \alpha + \cos \beta + \cos \gamma + 1 = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2}$$

$$= 2 \cos \left(\frac{\pi - \gamma}{2} \right) \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2}$$

$$= -2 \cos \frac{\gamma}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2}$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\gamma}{2} - \cos \frac{\alpha-\beta}{2} \right\}$$

$$= 2 \cos \frac{\gamma}{2} \left\{ \cos \left(\pi - \frac{\alpha+\beta}{2} \right) - \cos \frac{\alpha-\beta}{2} \right\}$$

$$= -2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha+\beta}{2} + \cos \frac{\alpha-\beta}{2} \right\}$$

$$= -4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$$

167. First side = $2 \cos 9A \cos A + 6 \cos 3A \cos A$

$$\begin{aligned} &= 2 \cos A \{ \cos 9A + 3 \cos 3A \} \\ &= 2 \cos A \{ (4 \cos^3 3A - 3 \cos 3A) + 3 \cos 3A \} \\ &= 8 \cos A \cos^3 3A. \end{aligned}$$

168. $r_1 + r_2 = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \frac{\Delta(s-a+s-b)}{(s-a)(s-b)} = c \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$

Also

$$r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2 \quad [\text{XVIII. } a. \text{ Ex. 24}];$$

$$\begin{aligned} \therefore \text{First side} &= \frac{abc}{4s^2} \sqrt{\frac{s(s-c)}{(s-a)(s-b)} \cdot \frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)}} \\ &= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\Delta} = R. \end{aligned}$$

169. $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = -\frac{2 \tan^2 \phi}{2(1 + \tan^2 \phi)}$
 $= -\sin^2 \phi.$

170. First side = $\frac{\sin A \sin (60^\circ + A) \sin (120^\circ + A)}{\cos A \cos (60^\circ + A) \cos (120^\circ + A)}$
 $= \frac{\sin A}{\cos A} \cdot \frac{\cos 60^\circ - \cos (180^\circ + 2A)}{\cos 60^\circ + \cos (180^\circ + 2A)}$
 $= \frac{\sin A}{\cos A} \cdot \frac{1 + 2 \cos 2A}{1 - 2 \cos 2A}$
 $= \frac{\sin A + 2 \sin A \cos 2A}{\cos A - 2 \cos A \cos 2A}$
 $= \frac{\sin A + (\sin 3A - \sin A)}{\cos A - (\cos 3A + \cos A)}$
 $= -\frac{\sin 3A}{\cos 3A} = -\tan 3A.$

171. We have $A - B = B$; therefore $\sin(A - B) = \sin B$;

also $\sin(A + B) = \sin(180^\circ - C) = \sin C$;

$$\therefore \sin(A + B) \sin(A - B) = \sin C \sin B;$$

that is,

$$\sin^2 A - \sin^2 B = \sin B \sin C;$$

or

$$a^2 - b^2 = bc.$$

172. If a, b be the sides of the triangle and square respectively, and R the radius of the circle, it is easy to shew that

$$a = R\sqrt{3}; \quad b = 2R \cos 45^\circ = R\sqrt{2}.$$

$$\begin{aligned}
 173. \text{ We have } & \frac{c+b}{c-b} \tan \frac{A}{2} = \frac{\sin C + \sin B}{\sin C - \sin B} \tan \frac{A}{2} \\
 &= \frac{\sin(A+B) + \sin B}{\sin(A+B) - \sin B} \tan \frac{A}{2} \\
 &= \frac{2 \sin\left(\frac{A}{2} + B\right) \cos \frac{A}{2}}{2 \cos\left(\frac{A}{2} + B\right) \sin \frac{A}{2}} \tan \frac{A}{2} \\
 &= \tan\left(\frac{A}{2} + B\right).
 \end{aligned}$$

$$\tan\left(\frac{A}{2} + B\right) = \frac{7b+3b}{7b-3b} \tan\frac{A}{2} = \frac{10}{4} \tan\frac{A}{2},$$

$$\begin{array}{r} \log 10 = 1 \\ \log 4 = .60206 \\ \hline & .3979400 \end{array}$$

$$\log \tan 3^\circ 18' 42'' = 2.7624069$$

$$\log \tan \left(\frac{A}{2} + B \right) = \overline{1.1603469}$$

$$\log \tan 8^\circ 13' 50'' = \frac{1.1603083}{386}$$

$$\text{prop'l increase} = \frac{386}{1486} \times 60''$$

= 2°6''

$$\therefore \frac{A}{2} + B = 8^\circ 13' 53'', \text{ and } \frac{A}{2} = 3^\circ 18' 42'';$$

$$\therefore B = 4^\circ 55' 11'',$$

$$C = 168^\circ 27' 25''.$$

174. By a well-known geometrical property, we have

$$AC^2 + AB^2 = 2AD^2 + 2DB^2.$$

$$\therefore AC^2 - AB^2 = 2(AD^2 + DB^2 - AB^2)$$

$$= 4AD \cdot DB \cos ADB$$

$$= 4AD \cdot DB \sin ADB \cot ADB$$

$$= (AD, DB \sin ADB) 4 \cot ADB$$

$$= 4\Delta \cot ADB,$$

for AD , $DB \sin ADB = 2$ (area of triangle ADB) = Δ .

$$\begin{aligned}175. \quad \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 40^\circ \sin 80^\circ}{2 \cos 40^\circ \cos 80^\circ} \\&= \frac{\cos 40^\circ - \cos 120^\circ}{\cos 40^\circ + \cos 120^\circ} \\&= \frac{2 \cos 40^\circ + 1}{2 \cos 40^\circ - 1};\end{aligned}$$

$$\begin{aligned}\therefore \tan 20^\circ \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 20^\circ \cos 40^\circ + \sin 20^\circ}{2 \cos 20^\circ \cos 40^\circ - \cos 20^\circ} \\&= \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ.\end{aligned}$$

176. See Art. 144.

$$\begin{aligned}177. \quad 2R &= \frac{abc}{2\Delta}; \quad 2r = \frac{2\Delta}{s}; \\&\therefore 2R \cdot 2r = \frac{abc}{s} = \frac{2abc}{a+b+c}.\end{aligned}$$

$$178. \quad \sin B = \frac{b}{a} \sin A = \frac{4\sqrt{3}}{7}$$

$$\frac{1}{2} \log 3 = .2385606$$

$$2 \log 2 = .6020600$$

$$\log 7 = .8450980$$

$$\log \sin B = \bar{1}.9955226$$

$$\log \sin 81^\circ 47' = \bar{1}.9955188$$

$$\qquad \qquad \qquad \text{prop'l. increase} = \frac{38}{183} \times 60''$$

$$\qquad \qquad \qquad = 12.4''.$$

$\therefore B = 81^\circ 47' 12''$; but since $a < b$ there is another value of B supplementary to this, viz. $98^\circ 12' 48''$.

$$\therefore C = 68^\circ 12' 48'', \text{ or } 51^\circ 47' 12''.$$

To find c , we have $c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0$,

[Art. 150]

that is

$$c^2 - 24c + 143 = 0,$$

$$(c - 13)(c - 11) = 0;$$

$$\therefore c = 13, \text{ or } 11.$$

$$\begin{aligned}
 179. \quad \tan^2\left(\frac{\pi}{4} + \beta\right) &= \frac{1 + \sin 2\beta}{1 - \sin 2\beta} & [\text{XI. f. Ex. 15}] \\
 &= \frac{1 + \sin 2\alpha \sin 2\alpha' + \sin 2\alpha + \sin 2\alpha'}{1 + \sin 2\alpha \sin 2\alpha' - \sin 2\alpha - \sin 2\alpha'} \\
 &= \frac{(1 + \sin 2\alpha)(1 + \sin 2\alpha')}{(1 - \sin 2\alpha)(1 - \sin 2\alpha')} \\
 &= \tan^2\left(\frac{\pi}{4} + \alpha\right) \tan^2\left(\frac{\pi}{4} + \alpha'\right).
 \end{aligned}$$

180. With the figure of Art. 199, let

$$PC = x, \beta = 28^\circ, \alpha = 16^\circ, a = 16071 \text{ feet.}$$

Then

$$x = \frac{16071 \sin 28^\circ \sin 16^\circ}{\sin 12^\circ}.$$

$$\log 16071 = 4 \cdot 2060$$

$$\log \sin 28^\circ = 1.6716$$

$$\log \sin 16^\circ = \overline{1} \cdot 4403 \\ \underline{\quad 3 \cdot 3179}$$

$$\log \sin 12^\circ = 1.3179$$

$$\log x = \overline{4.0000}$$

$x = 10000$ feet.

.. w = 20000 100 ..

$$C = A - C_{\text{min}}$$

$$\cos \frac{\theta}{2} = 2 \sin \phi$$

$$= 4 \sin$$

$$182. \text{ We have } 2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \sin (A+C)$$

$$= 4 \sin \frac{A+C}{2} \cos \frac{A+C}{2};$$

either

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$$\cos \frac{A-C}{2} = 2 \sin \frac{A+C}{2} \quad \dots \dots \dots \quad (2).$$

From (1) $\frac{A+C}{2} = (2n+1)\frac{\pi}{2}$, and from (2) by expanding each side and dividing throughout by $\cos \frac{A}{2} \cos \frac{C}{2}$ we obtain the other result.

$$183. \text{ First side} = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{C}{2} \right\}$$

$$\begin{aligned}
 &= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{(A+B)}{2} \right\} \\
 &= 2 \sin \frac{C}{2} \cdot 2 \sin \left(\frac{\pi}{4} - \frac{B}{2} \right) \sin \left(\frac{\pi}{4} - \frac{A}{2} \right) \\
 &= 4 \sin \frac{C}{2} \sin \left(45^\circ - \frac{A}{2} \right) \sin \left(45^\circ - \frac{B}{2} \right).
 \end{aligned}$$

184. $\frac{bc}{r_1} = \frac{bc(s-a)}{\Delta} = \frac{4R(s-a)}{a} = 2R \left(\frac{b+c-a}{a} \right);$

$$\begin{aligned}
 \therefore \text{first side} &= 2R \left(\frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c} \right) \\
 &= 2R \left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3 \right).
 \end{aligned}$$

185. Since the points A, B, E, C are concyclic, $\angle BED = \angle C$; also

$$\angle EBD = \angle EAC = \frac{A}{2};$$

$$\therefore \text{from } \Delta BDE, \quad BD = \frac{DE \sin C}{\sin \frac{A}{2}},$$

$$\text{from } \Delta DEC, \quad DC = \frac{DE \sin B}{\sin \frac{A}{2}};$$

$$\therefore \text{by addition } a = \frac{DE}{\sin \frac{A}{2}} (\sin B + \sin C);$$

$$\therefore a^2 = \frac{DE}{\sin \frac{A}{2}} (a \sin B + a \sin C)$$

$$= \frac{DE(b+c) \sin A}{\sin \frac{A}{2}};$$

$$\therefore DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}.$$

186. $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{5875.5 \times 1785.5}{3850 \times 3811}}$

log 5875.5 = 3.7690448	log 3850 = 3.5854607
log 1785.5 = 3.2517599	log 3811 = 3.5810389
7.0208047	7.1664996
7.1664996	
2 <u>1.8543051</u>	
log cos $\frac{A}{2}$ = 1.9271525	
log cos $32^\circ 16'$ = <u>1.9271509</u>	prop ^{l.} decrease = $\frac{16}{797} \times 60''$
16	$= 1.2'';$
$\therefore \frac{A}{2} = 32^\circ 15' 58.8'', \text{ or } A = 64^\circ 31' 58''.$	

188. See Art. 117.

190. (1) $\frac{b^2 + c^2 - a^2}{a^2 + c^2 - b^2} = \frac{2bc \cos A}{2ca \cos B} = \frac{b \cos A}{a \cos B}$
 $= \frac{\sin B \cos A}{\sin A \cos B} = \frac{\tan B}{\tan A}.$

(2) $\frac{2 \sin^2 A}{a^2} = \frac{2 \sin^2 B}{b^2};$
 $\therefore \frac{1 - \cos 2A}{a^2} = \frac{1 - \cos 2B}{b^2},$

that is, $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{b}{b^2}.$

191. $\Delta = \sqrt{(s-a)(s-b)(s-c)} = \sqrt{110.33.42.35}$
 $= \sqrt{11^2 \cdot 7^2 \cdot 3^2 \cdot 2^2 \cdot 5^2} = 11 \cdot 7 \cdot 3 \cdot 2 \cdot 5$
 $= 2310 \text{ sq. ft.}$

$$r_1 = \frac{\Delta}{s-a} = \frac{2310}{33} = 70 \text{ ft.}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{2310}{42} = 55 \text{ ft.}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{2310}{35} = 66 \text{ ft.}$$

192. Draw DK, DK' perpendicular to AB, AC respectively; then
 $DK \cdot AB + DK' \cdot AC = 2\Delta;$

that is, $c \cdot AD \sin \frac{A}{2} + b \cdot AD \sin \frac{A}{2} = bc \sin A;$

$$\therefore AD(b+c) = 2bc \cos \frac{A}{2}.$$

193. (1) $\sin 5\theta - \sin 3\theta = \sqrt{2} \sin \theta.$

$$\therefore 2 \sin \theta \cos 4\theta = \sqrt{2} \sin \theta;$$

that is,

$$\sin \theta = 0, \text{ or } \cos 4\theta = \frac{1}{\sqrt{2}}.$$

$$\therefore \theta = n\pi, \text{ or } 4\theta = 2n\pi \pm \frac{\pi}{4}.$$

(2) $\cot \theta + \cot \left(\frac{\pi}{4} + \theta \right) = 2;$

$$\therefore \cot \theta + \frac{\cot \theta - 1}{\cot \theta + 1} = 2;$$

$$\cot^2 \theta + 2 \cot \theta - 1 = 2 \cot \theta + 2;$$

$$\cot^2 \theta = 3;$$

that is,

$$\cot \theta = \pm \sqrt{3}, \text{ and } \theta = n\pi \pm \frac{\pi}{6}.$$

194. The given relation easily reduces to $\cos 2\alpha = \sin 2\beta$, one solution of which is $2\alpha = \frac{\pi}{2} - 2\beta$.

195. We have $\tan(\alpha + \theta) \tan(\alpha - \theta) = \tan^2 \beta;$

$$\therefore \frac{\tan^2 \alpha - \tan^2 \theta}{1 - \tan^2 \alpha \tan^2 \theta} = \tan^2 \beta;$$

whence

$$\tan^2 \theta (1 - \tan^2 \alpha \tan^2 \beta) = \tan^2 \alpha - \tan^2 \beta;$$

$$\begin{aligned} \therefore \tan^2 \theta &= \frac{(\tan \alpha + \tan \beta)(\tan \alpha - \tan \beta)}{(1 - \tan \alpha \tan \beta)(1 + \tan \alpha \tan \beta)} \\ &= \tan(\alpha + \beta) \tan(\alpha - \beta). \end{aligned}$$

196. (1) We have $p_1 = \frac{2\Delta}{a};$

\therefore second side $= \frac{a^3 b^3 c^3}{8\Delta^3} = 8 \left(\frac{abc}{4\Delta} \right)^3 = 8R^3.$

(2) Second side $= \frac{a^2 + b^2 - 2ab \cos C}{4\Delta^2}$

$$= \frac{c^2}{4\Delta^2} = \left(\frac{c}{2\Delta} \right)^2 = \frac{1}{p_3^2}.$$

197. By Art. 215 we have

$$\text{perimeter} = 30r \tan \frac{\pi}{15}, \text{ where } \pi r^2 = 1386.$$

Now

$$r^2 = \frac{7}{22} \times 1386 = 7 \times 63; \therefore r = 21;$$

$$\therefore \text{perimeter} = 30 \times 21 \tan 12^\circ$$

$$= 630 \times .213$$

$$= 134.19 \text{ ft.}$$

198. Let A, B represent the foot of the pole in the two positions; C, S the top of the pole on the coping and sill respectively; also let W be the foot of the wall.

Then

$$x + SW = AC \sin \alpha,$$

but

$$SW = BS \sin \beta = AC \sin \beta;$$

$$\therefore x = AC (\sin \alpha - \sin \beta).$$

Similarly

$$a = AC (\cos \beta - \cos \alpha);$$

$$\therefore \frac{x}{a} = \frac{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}} = \cot \frac{\alpha + \beta}{2}.$$

199. See Examples XVIII. *a.* 18. Each of the three expressions will be found to be equal to r .

200. (1) Let $\sin^{-1} \frac{2}{7} = \theta$; then $\cos 2\theta = 1 - 2 \sin^2 \theta = \frac{41}{49}$;

$$\therefore \cos^{-1} \frac{41}{49} = 2\theta = 2 \sin^{-1} \frac{2}{7}.$$

$$(2) 3 \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{3 \times \frac{1}{4} - \left(\frac{1}{4}\right)^3}{1 - 3 \left(\frac{1}{4}\right)^2}$$

$$= \tan^{-1} \frac{3 \times 16 - 1}{64 - 12} = \tan^{-1} \frac{47}{52}.$$

201. (1) As in XI. *f.* Ex. 14 we may prove that

$$\tan A + \sec A = \tan \left(45^\circ + \frac{A}{2}\right).$$

Also

$$\cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2};$$

$$\therefore (\tan A + \sec A) \cot \frac{A}{2} = (\cot A + \operatorname{cosec} A) \tan \left(45^\circ + \frac{A}{2}\right).$$

(2) First side

$$\begin{aligned} &= 2 \cos(A+B) \cos(A-B) - 2 \cos(A+B) \{\cos(A-B) - \cos(90^\circ - A+B)\} \\ &= 2 \cos(A+B) \sin(A+B) = \sin 2(A+B). \end{aligned}$$

202. By Art. 214, $\frac{\text{perimeter of fig.}}{\text{diameter of circle}} = 7 \sin 25\frac{5}{7}^\circ$,

$$\log 7 \sin 25\frac{5}{7}^\circ = .8450980 + 1.6373733$$

 $= .4824713,$ which is greater than $\log 3.$
203. We have $c^2 = a^2 + b^2 - 2ab \cos C$

$$\begin{aligned} &= (a^2 + b^2) \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) \\ &= (a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2} \\ &= (a+b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left(\frac{a-b}{a+b} \right)^2 \cot^2 \frac{C}{2} \right\} \\ &= (a+b)^2 \sin^2 \frac{C}{2} (1 + \tan^2 \phi) \\ &= (a+b)^2 \sin^2 \frac{C}{2} \sec^2 \phi; \\ \therefore c &= (a+b) \sin \frac{C}{2} \sec \phi. \end{aligned}$$

204. We have $\tan \phi = \frac{237 - 158}{237 + 158} \cot 33^\circ 20'$

$$= \frac{2}{10} \cot 33^\circ 20';$$

$$\begin{aligned} \therefore \log \tan \phi &= \log 2 - 1 + \log \cot 33^\circ 20' \\ &= 1.30103 + 1.8197 \\ &= 1.48300 \end{aligned}$$

$$\begin{aligned} \log \tan 16^\circ 54' &= \underline{1.48262} & \text{prop'l increase} &= \frac{38}{46} \times 60'' \\ \text{diff.} & \quad \underline{38} & &= 50''; \end{aligned}$$

$$\therefore \phi = 16^\circ 54' 50'',$$

$$\begin{aligned} \log \sec 16^\circ 55' &= .01921 & \text{prop'l increase for } 50'' &= \frac{4}{60} \times 50'' \\ \log \sec 16^\circ 54' &= .01917 & &= 3''; \\ \text{diff. for } 60'' & \quad \underline{4} & & \end{aligned}$$

$$\therefore \log \sec \phi = .01920.$$

Now $c = (a+b) \sin \frac{C}{2} \sec \phi = 395 \sin \frac{C}{2} \sec \phi,$

$$\begin{aligned}\log 395 &= \log 79 + 1 - \log 2 \\ &= 2.59660\end{aligned}$$

$$\log \sin \frac{C}{2} = 1.73998$$

$$\log \sec \phi = .01920$$

$$\log c = 2.35578$$

$$\therefore c = 226.87.$$

205. $2 \cos^2 2\theta = 1 + \cos 4\theta;$

$$\therefore 2 \cos 2\theta = \sqrt{2 + 2 \cos 4\theta}.$$

Similarly $2 \cos \theta = \sqrt{2 + 2 \cos 2\theta} = \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}.$

206. Let $\sin^{-1} \frac{3}{\sqrt{73}} = \alpha$, so that $\cos \alpha = \frac{8}{\sqrt{73}},$

and let $\cos^{-1} \frac{11}{\sqrt{146}} = \beta$, so that $\sin \beta = \frac{5}{\sqrt{146}}.$

Then $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \frac{3}{\sqrt{73}} \cdot \frac{11}{\sqrt{146}} + \frac{8}{\sqrt{73}} \cdot \frac{5}{\sqrt{146}} = \frac{73}{73\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \sin \frac{\pi}{4} = \sin \left(\frac{5\pi}{12} - \frac{\pi}{6} \right);$$

$$\therefore \alpha + \beta = \frac{5\pi}{12} - \frac{\pi}{6} = \frac{5\pi}{12} - \sin^{-1} \frac{1}{2}.$$

Again $\tan^{-1} \frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}} = \tan^{-1} \frac{23}{36};$

$$\tan^{-1} \frac{4x^2 - 2}{6x} = \tan^{-1} \frac{23}{36};$$

$$\therefore 36(2x^2 - 1) = 69x,$$

$$24x^2 - 23x - 12 = 0;$$

$$\therefore (3x - 4)(8x + 3) = 0,$$

that is, $x = \frac{4}{3}, \text{ or } -\frac{3}{8}.$

207. We have $x = \frac{2\Delta}{a}$, $R = \frac{abc}{4\Delta}$;

$$\therefore x = \frac{1}{a} \cdot \frac{abc}{2R}; \quad \therefore \frac{bx}{c} = \frac{b^2}{2R},$$

that is,

$$\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{2R}.$$

208. We have $\cos(\alpha + \theta) = \cos \left\{ \frac{\pi}{2} - (\alpha - \theta) \right\}$,

$$\therefore \alpha + \theta = 2m\pi \pm \left\{ \frac{\pi}{2} - (\alpha - \theta) \right\};$$

the upper sign gives

$$2\alpha = 2m\pi + \frac{\pi}{2},$$

and the lower sign gives $2\theta = 2m\pi - \frac{\pi}{2}$.

209. With the notation of Art. 228,

$$SI^2 = R^2 - 2Rr.$$

If a be the base of the triangle, $A = 120^\circ$,

$$B = C = 30^\circ; \quad \therefore r = 4R \sin 60^\circ \sin 15^\circ \sin 15^\circ;$$

$$\therefore SI^2 = R^2 - 8R^2 \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3}-1}{2\sqrt{2}} \right)^2$$

$$= R^2 (4 - 2\sqrt{3});$$

$$\therefore SI = R(\sqrt{3} - 1)$$

$$= \frac{a}{2 \sin A} (\sqrt{3} - 1) = \frac{a(\sqrt{3} - 1)}{\sqrt{3}},$$

$$\therefore SI : a = \sqrt{3} - 1 : \sqrt{3}.$$

210. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$$= 1 + \frac{r}{R} = 1 + \frac{3}{4},$$

$$\therefore 4(\cos A + \cos B + \cos C) = 7.$$

211. Since $\frac{\cos(\theta - \alpha)}{\sin(\theta + \alpha)} = \frac{1+m}{1-m}$,

dividendo and componendo, we have

$$\frac{\cos(\theta - \alpha) - \sin(\theta + \alpha)}{\cos(\theta - \alpha) + \sin(\theta + \alpha)} = m,$$

By expanding the sines and cosines we obtain

$$\frac{(\cos \theta - \sin \theta)(\cos \alpha - \sin \alpha)}{(\cos \theta + \sin \theta)(\cos \alpha + \sin \alpha)} = m,$$

or $\frac{1 - \tan \theta}{1 + \tan \theta} = m \left(\frac{\cot \alpha + 1}{\cot \alpha - 1} \right).$ [See XI. b. Ex. 6, 7.]

212. (1) $\sin 5\theta - \sin 3\theta = \sqrt{2} \cos 4\theta;$

$$2 \cos 4\theta \sin \theta = \sqrt{2} \cos 4\theta;$$

$$\therefore \cos 4\theta = 0; \text{ whence } 4\theta = (2n+1) \frac{\pi}{2},$$

or $\sin \theta = \frac{1}{\sqrt{2}}; \text{ whence } \theta = n\pi + (-1)^n \frac{\pi}{4}.$

(2) $1 + \sin 2\theta = \frac{1 + \tan \theta}{1 - \tan \theta}.$

$$1 + \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{1 + \tan \theta}{1 - \tan \theta}.$$

$$(1 + \tan \theta)^2 (1 - \tan \theta) = (1 + \tan \theta) (1 + \tan^2 \theta);$$

$$\therefore 1 + \tan \theta = 0; \text{ whence } \theta = n\pi + \frac{3\pi}{4},$$

or $1 - \tan^2 \theta = 1 + \tan^2 \theta;$

$$\therefore \tan \theta = 0; \text{ whence } \theta = n\pi.$$

213. We have $2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2};$

$$\therefore \cos \frac{A-B}{2} = 2 \sin \frac{C}{2}, \text{ or } 2 \cos \frac{A-B}{2} \sin \frac{A+B}{2} = 4 \sin \frac{C}{2} \cos \frac{C}{2};$$

that is, $\sin A + \sin B = 2 \sin C, \text{ or } a + b = 2c.$

214. With the figure on p. 186, we have $\tan \beta = \frac{1}{9}$, $PA = 80 \text{ ft.}$, $CA = 100 \text{ ft.}$. Let $BP = x \text{ ft.}$, then

$$\tan \theta = \frac{x+80}{100}, \quad \tan (\theta - \beta) = \frac{80}{100} = \frac{4}{5},$$

$$\therefore \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta} = \frac{4}{5},$$

$$\frac{\frac{x+80}{100} - \frac{1}{9}}{1 + \frac{x+80}{900}} = \frac{4}{5};$$

$$\therefore 5(9x + 720 - 100) = 4(980 + x);$$

$$45x + 3100 = 3920 + 4x;$$

$$41x = 820; \text{ or } x = 20.$$

$$215. \quad (1) \quad \cot^{-1} 7 + \cot^{-1} 8 = \cot^{-1} \frac{7 \cdot 8 - 1}{7+8} = \cot^{-1} \frac{55}{15},$$

$$\cot^{-1} 3 - \cot^{-1} 18 = \cot^{-1} \frac{3 \cdot 18 + 1}{18-3} = \cot^{-1} \frac{55}{15}.$$

$$(2) \quad 4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{5^2}} = 2 \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \frac{\frac{2 \times 5}{12}}{1 - \frac{5^2}{12^2}} = \tan^{-1} \frac{120}{119}.$$

$$\therefore 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \tan^{-1} \frac{120 \cdot 239 - 119}{119 \cdot 239 + 120}$$

$$= \tan^{-1} \frac{119 \cdot 239 + (239 - 119)}{119 \cdot 239 + 120} = \tan^{-1} 1 = \frac{\pi}{4}.$$

216. Proceeding as in Art. 259, Ex. 2, we find that $\frac{A}{2}$ lies between $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$; that is A lies between $(8n+3)\frac{\pi}{2}$ and $(8n+5)\frac{\pi}{2}$.

$$217. \quad \text{First side} = \frac{1}{2} \left\{ 1 - \cos \left(\frac{\pi}{4} + \theta \right) - 1 + \cos \left(\frac{\pi}{4} - \theta \right) \right\}$$

$$= \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} \sin \theta.$$

218. From the two given relations we easily deduce

$$x = \frac{\sin \theta}{\sin (\theta + \phi)}, \quad y = \frac{\sin \phi}{\sin (\theta + \phi)},$$

$$\therefore \sin \theta : \sin \phi = x : y.$$

$$219. \quad \tan^{-1} \frac{x+1+x-1}{1-(x^2-1)} = \tan^{-1} \frac{8}{31};$$

$$\therefore \frac{2x}{2-x^2} = \frac{8}{31}, \text{ or } 4x^2 + 31x - 8 = 0;$$

$$\therefore (4x-1)(x+8)=0, \text{ or } x = \frac{1}{4}, \text{ or } -8.$$

$$\begin{aligned}\text{Again } \sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) \\ &= 1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3) \\ &= 1 + 4 + 1 + 9 = 15.\end{aligned}$$

220. Let $5A = \alpha$, $5B = \beta$, $5C = \gamma$, then $\alpha + \beta + \gamma = 5\pi$;

$$\begin{aligned}\therefore \sin(\alpha + \beta) &= \sin(\pi - \gamma) = \sin \gamma; \cos \gamma = -\cos(\alpha + \beta), \\ \sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2 \sin(\alpha + \beta) \cos(\alpha - \beta) + 2 \sin \gamma \cos \gamma \\ &= 2 \sin \gamma \{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} \\ &= 4 \sin \alpha \sin \beta \sin \gamma.\end{aligned}$$

In the second case the sum of the three angles is $\frac{16\pi}{25}$, or $\frac{\pi}{2}$ and the result easily follows as in Art. 135, Ex. 2.

221. See solution to XVIII. a. Ex. 24.

$$222. \text{ We have } x = \frac{BD \sin 15^\circ}{\sin 50^\circ}; \quad BD = \frac{100}{\cos 25^\circ};$$

$$\therefore x = \frac{100 \cos 75^\circ}{\cos 40^\circ \cos 25^\circ};$$

$$\begin{aligned}\therefore \log x &= 2 + \log \cos 75^\circ - (\log \cos 40^\circ + \log \cos 25^\circ) \\ &= 1.4129962 - (1.8415297)\end{aligned}$$

$$\begin{aligned}&= 1.5714665 && \text{prop'l. increase} = \frac{22}{116} \times .001 \\ \log 37.279 &= \frac{1.5714643}{22} && = .00019; \\ \text{diff.} &&&\end{aligned}$$

$$\therefore x = 37.27919.$$

$$\begin{aligned}223. \quad \{\sec \theta + \operatorname{cosec} \theta (1 + \sec \theta)\}^2 &= \left(\frac{1}{\cos \theta} + \frac{1 + \cos \theta}{\sin \theta \cos \theta} \right)^2 \\ &= \frac{(1 + \sin \theta + \cos \theta)^2}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{2 + 2 \sin \theta + 2 \cos \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta \cos^2 \theta};\end{aligned}$$

$$\begin{aligned}\therefore \text{First side} &= 2 \sec^2 \theta \frac{(1 - \cos \theta)}{\sin^2 \theta} (1 + \sin \theta + \cos \theta + \sin \theta \cos \theta) \\ &= \frac{2 \sec^2 \theta (1 + \cos \theta) (1 + \sin \theta)}{1 + \cos \theta} \\ &= 2 \sec^2 \theta (1 + \sin \theta).\end{aligned}$$

224. The relation given will be true if

$$\frac{1}{a+b+c} - \frac{1}{a+c} = \frac{1}{b+c} - \frac{2}{a+b+c};$$

i.e. if $\frac{b}{a+c} = \frac{a-b-c}{b+c}$, or $\frac{b}{a+c} = \frac{b+c-a}{b+c}$,

i.e. if $b(b+c) = (b+c)(a+c) - a^2 - ac$.

From this we easily deduce $\frac{a^2 + b^2 - c^2}{ab} = 1$, which is true when $C = 60^\circ$.

225. The solution of this example is merely an extension of that of Ex. 205.

226. We have $m \sin(\alpha - \theta) \cos(\alpha - \theta) = n \sin \theta \cos \theta$,

$$m \sin 2(\alpha - \theta) = n \sin 2\theta;$$

$$\therefore \frac{\sin 2(\alpha - \theta) - \sin 2\theta}{\sin 2(\alpha - \theta) + \sin 2\theta} = \frac{n-m}{n+m},$$

$$\frac{\sin(\alpha - 2\theta) \cos \alpha}{\cos(\alpha - 2\theta) \sin \alpha} = \frac{n-m}{n+m},$$

$$\tan(\alpha - 2\theta) = \frac{n-m}{n+m} \tan \alpha,$$

$$\alpha - 2\theta = \tan^{-1} \left(\frac{n-m}{n+m} \tan \alpha \right),$$

$$\theta = \frac{1}{2} \left\{ \alpha - \tan^{-1} \left(\frac{n-m}{n+m} \tan \alpha \right) \right\}.$$

227. Put k for each of the equal ratios, then it easily follows that $s = k(1+n^2)$.

Now $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(1-m^2)m^2(1+n^2)}{(1+n^2)n^2(1-m^2)}} = \frac{m}{n};$

$$\therefore A = 2 \tan^{-1} \frac{m}{n}; \text{ similarly } B = 2 \tan^{-1} mn.$$

Again,

$$\begin{aligned}\Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= k^2 \sqrt{(1+n^2)^2 (1-m^2)^2 m^2 n^2} \\ &= k^2 (1-m^2) (1+n^2) mn = kcmn \\ &= \frac{mnbc}{m^2+n^2}, \text{ since } \frac{b}{m^2+n^2} = k.\end{aligned}$$

228. See figure and solution of Example II. page 190.

Here

$$h = CD = \frac{a \sin \beta}{\cos(2\alpha + \beta)},$$

$$l = DE = \frac{a \sin \alpha \cos(\alpha + \beta)}{\cos(2\alpha + \beta)}.$$

But $2\alpha + \beta + \theta = 90^\circ$; $\therefore \cos(2\alpha + \beta) = \sin \theta$;

\therefore by substitution, $h = a \sin \beta \operatorname{cosec} \theta$,

$$\begin{aligned} 2l &= 2a \operatorname{cosec} \theta \sin \alpha \cos(\alpha + \beta) \\ &= a \operatorname{cosec} \theta \{ \sin(2\alpha + \beta) - \sin \beta \} \\ &= a \operatorname{cosec} \theta (\cos \theta - \sin \beta). \end{aligned}$$

229. We have

$$\begin{aligned} \frac{1}{2} \tan \frac{\theta}{2} + \cot \theta \\ &= \frac{1}{2} \cdot \frac{1 - \cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta + 2 \cos \theta}{2 \sin \theta} = \frac{1 + \cos \theta}{2 \sin \theta} \\ &= \frac{2 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\ &= \frac{\cos^2 \frac{\theta}{4} - \sin^2 \frac{\theta}{4}}{4 \sin \frac{\theta}{4} \cos \frac{\theta}{4}} = \frac{1}{4} \cot \frac{\theta}{4} - \frac{1}{4} \tan \frac{\theta}{4}. \end{aligned}$$

230. With the figure of Art. 214 we have $AD = \frac{AO}{2m}$.

Let $\theta = \angle AOD$, then $\cos AOB = 1 - 2 \sin^2 \theta$

$$= 1 - \frac{2AD^2}{AO^2} = 1 - \frac{1}{2m^2} = \frac{2m^2 - 1}{2m^2};$$

$$\therefore AOB = \sec^{-1} \frac{2m^2}{2m^2 - 1}.$$

231. With the figure of Art 268, Ex. 1, we have

$$\frac{PN}{ON} = \tan 1'' = \text{radian measure of } 1'', \text{ approx.}$$

$$= \frac{\pi}{180} \times \frac{1}{60} \times \frac{1}{60};$$

$$\begin{aligned} \therefore ON &= \frac{180 \times 60 \times 60}{\pi} \text{ inches} \\ &= \frac{180 \times 60 \times 60}{1760 \times 3 \times 12\pi} \text{ miles} \\ &= \frac{1800}{176\pi} \text{ miles} = 3\frac{1}{4} \text{ miles, nearly.} \end{aligned}$$

$$\begin{aligned}
 232. \quad \tan^{-1} y &= 2 \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{\frac{4x}{1-x^2}}{1-\frac{4x^2}{(1-x^2)^2}} \\
 &= \tan^{-1} \frac{4x(1-x^2)}{1-6x^2+x^4}; \\
 \therefore y &= \frac{4x(1-x^2)}{1-6x^2+x^4}.
 \end{aligned}$$

$$\text{If } y = \tan \frac{\pi}{2}, \quad 1 - 6x^2 + x^4 = 0,$$

$$x = \tan \frac{1}{4} (\tan^{-1} y) = \tan \frac{\pi}{8},$$

thus $\tan \frac{\pi}{8}$ is a root of $x^4 - 6x^2 + 1 = 0$.

$$233. \text{ We have } \tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} = \frac{49}{529};$$

$$\therefore \tan a = \pm \frac{7}{23}.$$

The two values may be explained as in Art. 261, Ex. 2.

$$234. \text{ We have } \frac{\sin \theta}{a} = \frac{\sin \phi}{b} = \frac{\sin (\theta + \phi)}{c};$$

$$\frac{\sin \theta + \sin \phi}{a+b} = \frac{\sin (\theta + \phi)}{c}.$$

But

$$a+b=2c,$$

$$\therefore \sin \theta + \sin \phi = 2 \sin (\theta + \phi),$$

whence

$$\cos \frac{\theta - \phi}{2} = 2 \cos \frac{\theta + \phi}{2} \dots \dots \dots (1),$$

or

$$\begin{aligned}\cos \frac{\theta + \phi}{2} &= \cos \frac{\theta - \phi}{2} - \cos \frac{\theta + \phi}{2} \\ &= 2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \dots \dots \dots \quad (2)\end{aligned}$$

$$\begin{aligned}
 \text{Now } \cos \theta + \cos \phi &= 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\
 &= 2 \left(2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \right) 2 \cos \frac{\theta + \phi}{2}, \text{ by (1) and (2)} \\
 &= 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \cdot 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2}, \text{ by (2)} \\
 &= 16 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2} \\
 &= 4(1 - \cos \theta)(1 - \cos \phi).
 \end{aligned}$$

235. (1) $\sin 7\theta + \sin \theta = \sin 4\theta;$
 $\therefore 2 \sin 4\theta \cos 3\theta = \sin 4\theta;$
 $\therefore \text{either } \sin 4\theta = 0, \text{ or } \cos 3\theta = \frac{1}{2};$

that is, $4\theta = n\pi, \text{ or } 3\theta = 2n\pi \pm \frac{\pi}{3}.$

$$\begin{aligned}
 (2) \quad \tan x - \frac{\sqrt{3}}{\tan x} + 1 - \sqrt{3} &= 0; \\
 \tan^2 x - (\sqrt{3} - 1) \tan x - \sqrt{3} &= 0; \\
 (\tan x - \sqrt{3})(\tan x + 1) &= 0; \\
 \therefore \text{either } \tan x = \sqrt{3}, \text{ or } \tan x = -1;
 \end{aligned}$$

that is, $x = n\pi + \frac{\pi}{3}, \text{ or } x = n\pi + \frac{3\pi}{4}.$

236. (1) $\sin 3A = \sin 3(180^\circ - \overline{B+C})$
 $= \sin(360^\circ + 180^\circ - 3\overline{B+C}) = \sin 3\overline{B+C}.$

We have only now to prove that

$$\Sigma \sin 3(B+C) \sin(B-C) = 0,$$

and this follows by separating each term into the difference of two cosines.

(2) It will be sufficient to prove that

$$\Sigma \sin^3 A \sin(B-C) = 0.$$

Now $\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A;$

$$\begin{aligned}
 \therefore \Sigma \sin^3 A \sin(B-C) &= \frac{3}{4} \Sigma \sin A \sin \overline{B-C} - \frac{1}{4} \Sigma \sin 3A \sin \overline{B-C} \\
 &= 0,
 \end{aligned}$$

by the first part of the question.

237. The angle $ACP = \theta - A$.

$$\therefore \frac{AP}{PC} = \frac{\sin(\theta - A)}{\sin A} = \sin \theta \cot A - \cos \theta,$$

$$\frac{PC}{PB} = \frac{\sin B}{\sin(\theta+B)} = \frac{1}{\sin \theta \cot B + \cos \theta};$$

∴ by multiplication

$$\frac{AP}{PB} = \frac{m}{n} = \frac{\sin \theta \cot A - \cos \theta}{\sin \theta \cot B + \cos \theta};$$

whence

$$\sin \theta (n \cot A - m \cot B) = (m+n) \cos \theta,$$

or

$$(m+n) \cot \theta = n \cot A - m \cot B.$$

238. The equation may be written

Since α and β are roots of this equation

$$a \sin \alpha - \cos \alpha + b = 0,$$

$$a \sin \beta - \cos \beta + b = 0,$$

whence a and b may be found.

Again from (1),

$$(a \sin \theta + b)^2 = 1 - \sin^2 \theta.$$

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$$(1+a^2) \sin^2 \theta + 2ab \sin \theta + b^2 - 1 = 0;$$

since α , β are roots of this equation,

$$\sin \alpha + \sin \beta = -\frac{2ab}{1+a^2}.$$

Similarly we may shew that $\cos \alpha + \cos \beta = \frac{2b}{1+a^2}$, whence the required result follows.

239. Write s and c for $\sin \theta$ and $\cos \theta$ respectively; then

$$\frac{u_3 - u_5}{u_1} = \frac{s^3 + c^3 - (s^5 + c^5)}{s + c} = \frac{s^3(1 - s^2) + c^3(1 - c^2)}{s + c} = \frac{s^3c^2 + c^3s^2}{s + c} = s^2c^2.$$

$$\text{Again } \frac{u_5 - u_7}{u_3} = \frac{s^5 + c^5 - (s^7 + c^7)}{s^3 + c^3} = \frac{s^5(1 - s^2) + c^5(1 - c^2)}{s^3 + c^3}$$

$$= \frac{s^5c^2 + c^5s^2}{s^3 + c^3} = s^2c^2.$$

240. Let E, F be the first and second points of observation respectively; then $EF = a$, and EAD is a straight line. Let x = a side of the square base, then $EA = AB = AD = x$. Then if $\angle AFD = \theta$, we have

But $AD^2 = x^2$, $AF^2 = x^2 + a^2$, $FD^2 = 4x^2 + a^2$. Also $\cos \theta = \frac{1}{3}\sqrt{8}$. Substituting these values in (1) we obtain $x = \frac{a\sqrt{2}}{2}$.

$$241. \quad (1) \quad \text{First side} = \frac{1 - \sin^2 A}{\sin A} \cdot \frac{1 - \cos^2 A}{\cos A} = \frac{\cos^2 A \sin^2 A}{\sin A \cos A} \\ = \sin A \cos A.$$

$$\text{Again, second side} = \left(\frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \right)^{-1} = \sin A \cos A.$$

$$\begin{aligned}
 (2) \quad \text{First side} &= \frac{\tan \theta}{\sec^4 \theta} + \frac{\cot \theta}{\operatorname{cosec}^4 \theta} \\
 &= \sin \theta \cos^3 \theta + \cos \theta \sin^3 \theta \\
 &= \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) \\
 &= \frac{1}{2} \sin 2\theta.
 \end{aligned}$$

$$242. \text{ We have } 2 \sin 4\theta \cos \theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 5\theta + \sin 3\theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 3\theta = \frac{1}{2},$$

one solution of which is $3\theta = 30^\circ$.

$$243. \quad \tan^{-1} \frac{2mn}{m^2 - n^2} = \tan^{-1} \frac{\frac{2n}{m}}{1 - \frac{n^2}{m^2}} = 2 \tan^{-1} \frac{n}{m};$$

$$\begin{aligned}\therefore \text{first side} &= 2 \tan^{-1} \frac{n}{m} + 2 \tan^{-1} \frac{q}{p} \\ &= 2 \left(\tan^{-1} \frac{n}{m} + \tan^{-1} \frac{q}{p} \right) \\ &= 2 \tan^{-1} \frac{\frac{n}{m} + \frac{q}{p}}{1 - \frac{nq}{mp}};\end{aligned}$$

$$\therefore \text{first side} = 2 \tan^{-1} \frac{np + mq}{mp - nq} = 2 \tan^{-1} \frac{N}{M}$$

$$= \tan^{-1} \frac{2MN}{M^2 - N^2}.$$

244. Write $\frac{r}{s-a}$ for $\tan \frac{A}{2}$, then the first side becomes

$$\frac{r}{(s-a)(a-b)(a-c)} + \text{two similar terms.}$$

Now

$$\frac{r}{(s-a)(a-b)(a-c)} = -\frac{rs(b-c)(s-b)(s-c)}{\Delta^2(a-b)(b-c)(c-a)}$$

$$= -\frac{1}{\Delta} \frac{(b-c)\{s^2 - s(b+c) + bc\}}{(a-b)(b-c)(c-a)}.$$

Now $\Sigma(b-c)\{s^2 - s(b+c) + bc\} = -(a-b)(b-c)(c-a).$

[See Hall and Knight's Elem. Algebra, Art. 224.]

Thus the first side reduces to $\frac{1}{\Delta}$.

245. We have $\left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2;$

or $(b-a)(c-a) = 2(s-a)^2;$

$$bc - ac - ab + a^2 = 2s^2 - 4as + 2a^2;$$

$$\therefore bc = 2s^2 - 4as + a(a+b+c)$$

$$= 2s^2 - 4as + 2as;$$

whence

$$\frac{s(s-a)}{bc} = \frac{1}{2};$$

that is,

$$\cos \frac{A}{2} = \frac{1}{\sqrt{2}}, \text{ or } A = 90^\circ.$$

246. Let $A = 58^\circ 40' 3.9''$, $b = 237$, $c = 158$.

Then as in Art. 197 we obtain

$$a = (b+c) \sin \theta, \text{ where } \cos \theta = \frac{2\sqrt{bc}}{b+c} \cos \frac{A}{2},$$

$$\cos \theta = \frac{2\sqrt{237 \times 158}}{395} \cos 29^\circ 20' 1.95''$$

$$= \frac{2\sqrt{6}}{5} \cos 29^\circ 20' 1.95'',$$

$$\begin{array}{rcl}
 \log 2 & = & .3010300 \\
 \log 3 & = & .4771213 \\
 & & 2 | \underline{.7781513} \\
 \frac{1}{2} \log 6 & = & .3890757 \\
 2 \log 2 - 1 & = & \bar{1}.6020600 \\
 \log \cos 29^\circ 20' & = & \bar{1}.9404091 \\
 \text{Subtract diff. for } 1.95'' & & 23 \\
 \log \cos \theta & = & \bar{1}.9315425 \\
 \log \cos 31^\circ 20' & = & \bar{1}.9315374 \\
 & & \underline{51} \\
 & & \frac{51}{769} \times 60'' = 3.97''.
 \end{array}$$

$$\therefore \theta = 31^\circ 19' 56''.$$

Again

$$\begin{array}{rcl}
 \log 395 & = & 2.5965971 \\
 \log \sin 31^\circ 19' & = & \bar{1}.7158092 \\
 \text{diff. for } 56'' & & 1937 \\
 \log a & = & \underline{2.3126000} \\
 \log 205.4 & = & \underline{2.3126004} \\
 \therefore a & = & 205.4.
 \end{array}$$

$$247. \text{ We have } \frac{\sin(A+B)}{\cos(A+B)} = \frac{3 \sin A}{\cos A};$$

whence.

$$\sin(A+B)\cos A = 3 \cos(A+B)\sin A,$$

or

$$\sin(2A+B) + \sin B = 3 \sin(2A+B) - 3 \sin B;$$

that is

$$2 \sin(2A+B) = 4 \sin B.$$

Multiply by $\cos B$; then by separating the product on the left into the sum of two sines we obtain the required result.

$$\begin{aligned}
 248. \text{ First side} &= 2 \sin(\theta - a) \{ \sin(\theta - a) + \sin(2m\theta - a - \theta) \} \\
 &= 2 \sin^2(\theta - a) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2a) \\
 &= 1 - \cos(2\theta - 2a) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2a).
 \end{aligned}$$

249. Let AD be perpendicular to BC and meet the circum-circle in E ; then $\angle BED = C$, and $a = DE$.

$$\text{Now } \frac{BD}{a} = \tan C, \text{ and } \frac{DC}{a} = \tan B;$$

$$\therefore \frac{BD+DC}{a} = \frac{a}{a} = \tan B + \tan C.$$

$$\text{Similarly } \frac{b}{\beta} = \tan C + \tan A, \quad \frac{c}{\gamma} = \tan A + \tan B;$$

whence the result follows.

250. From the first equation, $3 \sin^2 A = 1 - 2 \sin^2 B - \cos 2B$; and from the second equation,

$$6 \sin A \cos A - 2 \sin 2B = 0;$$

multiply each term by $\sin A$, and we have

$$3 \sin^2 A \cos A - \sin 2B \sin A = 0.$$

Substituting $\cos 2B$ for $3 \sin^2 A$, we obtain

$$\cos 2B \cos A - \sin 2B \sin A = 0;$$

that is,

$$\cos(A + 2B) = 0; \text{ or } A + 2B = 90^\circ.$$

$$251. (1) \text{ First side} = \cot^{-1} \left(\frac{1}{\cot 2x} \right) + \cot^{-1} \left(-\frac{1}{\cot 3x} \right)$$

$$= \cot^{-1} \frac{\frac{1}{\cot 2x} \left(-\frac{1}{\cot 3x} \right) - 1}{-\frac{1}{\cot 3x} + \frac{1}{\cot 2x}}$$

$$= \cot^{-1} \left(\frac{\cot 3x \cot 2x + 1}{\cot 2x - \cot 3x} \right)$$

$$= \cot^{-1} (\cot x) = x.$$

$$(2) \text{ First side} = \tan^{-1} \frac{\frac{1-x}{1+x} - \frac{1-y}{1+y}}{1 + \frac{(1-x)(1-y)}{(1+x)(1+y)}} = \tan^{-1} \frac{2(y-x)}{2(1+xy)}$$

$$= \sin^{-1} \frac{\frac{y-x}{1+xy}}{\sqrt{1 + \frac{(y-x)^2}{(1+xy)^2}}} = \sin^{-1} \frac{y-x}{\sqrt{1+x^2} \sqrt{1+y^2}}.$$

252. See Art. 197.

Let A be the position of the station, B and C the positions of the two points; then $A = 49^\circ 45'$, $c = 1250$ yds., $b = 1575$ yds.

$$\text{Now } a = 2825 \cos \theta, \text{ where } \sin \theta = \frac{2 \sqrt{1250 \times 1575}}{2825} \cos 24^\circ 52' 30'',$$

$\log 1250$	$= 3.0969100$	$\log \cos 24^\circ 52' = \bar{1}.9577456$
$\log 1575$	$= 3.1972806$	$subtract \frac{1}{2} \times 586 \quad \underline{\hspace{1cm}} \quad \bar{1}.9577163$
	$2 6.2941906$	
	$\overline{3.1470953}$	
$\log 2$	$= .3010300$	
$\log \cos 24^\circ 52' 30''$	$= \bar{1}.9577163$	
	$\overline{3.4058416}$	
$\log 2825$	$= 3.4510185$	
$\log \sin \theta$	$= \bar{1}.9548231$	$= \log \sin 64^\circ 19'.$

Again

$$\begin{array}{rcl}
 \log 2825 & = & 3.4510185 \\
 \log \cos 64^\circ 19' & = & \overline{1.6368859} \\
 \log a & = & \overline{3.0879044} \\
 \log 1224.3 & = & \overline{3.0878878} \\
 & & \underline{166} \\
 & 4 & \underline{142} \\
 & & \underline{240} \\
 & 7 & \underline{249}
 \end{array}$$

$$\therefore a = 1224.347 \text{ yards.}$$

254. Multiply all through by 2; then

$$\begin{aligned}
 \text{First side} &= 1 + \cos 2S + 1 + \cos 2(S - A) + \text{two similar terms} \\
 &= 4 + 2 \cos(2S - A) \cos A + 2 \cos(2S - B - C) \cos(B - C) \\
 &= 4 + 2 \cos(B + C) \cos A + 2 \cos A \cos(B - C) \\
 &= 4 + 4 \cos A \cos B \cos C.
 \end{aligned}$$

255. It is easy to see that this is the same as Example 1 in Art. 135.

256. We have $R = \frac{a}{2 \sin A} = 18 \operatorname{cosec} 61^\circ 15'$.

$$\begin{array}{rcl}
 \log 18 & = & 1.2552725 \\
 \log \operatorname{cosec} 61^\circ 15' & = & \overline{.0571357} \\
 \log R & = & \overline{1.3124082} \\
 \log 20.530 & = & \overline{1.3123889} \\
 & & \underline{193} \\
 & 9 & \underline{191}
 \end{array}$$

$$\therefore R = 20.5309.$$

Again,

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$\begin{array}{rcl}
 \log R & = & 1.3124082 \\
 \log 4 & = & .6020600 \\
 \log \sin 30^\circ 37' & = & \overline{1.7069667} \\
 \text{diff. for } 30'' & = & 1067 \\
 \log \sin 36^\circ 37' & = & \overline{1.7755801} \\
 \text{diff. for } 30'' & = & 850 \\
 \log \sin 22^\circ 45' & = & \overline{1.5873865} \\
 \therefore \log r & = & \overline{.9845932} \\
 \log 9.6514 & = & \overline{.9845903} \\
 & & \underline{29} \\
 & 6 & \underline{27}
 \end{array}$$

$$\therefore r = 9.65146.$$

257. This follows from XVIII. c. Ex. 5 and XII. d. Ex. 12.

258. Let $\angle APB = \alpha$, $\angle BPC = \beta$, $\angle PBC = \gamma$;

$$\text{then } \frac{PB}{AB} = \frac{\sin(\gamma - \alpha)}{\sin \alpha}, \quad \frac{PB}{BC} = \frac{\sin(\beta + \gamma)}{\sin \beta};$$

but

$$AB = BC,$$

$$\therefore \frac{\sin(\gamma - \alpha)}{\sin \alpha} = \frac{\sin(\beta + \gamma)}{\sin \beta},$$

$$\text{or } \sin \gamma \cot \alpha - \cos \gamma = \cos \gamma + \sin \gamma \cot \beta;$$

$$\therefore 2 \cos \gamma = \sin \gamma (\cot \alpha - \cot \beta),$$

$$2 \cot \gamma = \cot \alpha - \cot \beta;$$

that is,

$$\frac{2}{T} = \frac{1}{t'} - \frac{1}{t},$$

since γ is the supplement of the angle BP makes with the road.

$$\begin{aligned} 259. \text{ First side} &= \frac{(\cos B + \cos C)(1 + 2 \cos A)}{(1 + 2 \cos A)(1 - \cos A)} = \frac{2 \cos \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin^2 \frac{A}{2}} \\ &= \frac{2 \cos \frac{B-C}{2}}{2 \sin \frac{A}{2}} \cdot \frac{\cos \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \cos \frac{B-C}{2} \sin \frac{B+C}{2}}{\sin A} \\ &= \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a}. \end{aligned}$$

$$260. \text{ From the fig. of Art. 219, we have } \frac{AI}{AI_1} = \frac{s-a}{s};$$

$$\therefore \text{first side} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-2s}{s} = 1$$

$$261. \text{ Let } \sin^{-1} \frac{1}{3} = \alpha, \text{ then } \cos \alpha = \frac{2\sqrt{2}}{3};$$

$$\sin^{-1} \frac{3}{\sqrt{11}} = \beta, \text{ then } \cos \beta = \frac{\sqrt{2}}{\sqrt{11}}.$$

$$\begin{aligned} \text{Now } \sin(\alpha + \beta) &= \frac{1}{3} \cdot \frac{\sqrt{2}}{\sqrt{11}} + \frac{2\sqrt{2}}{3} \cdot \frac{3}{\sqrt{11}} = \frac{7\sqrt{2}}{3\sqrt{11}} \\ &= \cos \left(\sin^{-1} \frac{1}{3\sqrt{11}} \right), \end{aligned}$$

$$\therefore \alpha + \beta = \frac{\pi}{2} - \sin^{-1} \frac{1}{3\sqrt{11}}.$$

$$\text{or } \sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{3\sqrt{11}} + \sin^{-1} \frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

262. We have $\cot \alpha (\cot \beta \cot \gamma - 1) = \cot \beta + \cot \gamma$,

$$\therefore \cot(\beta + \gamma) = \frac{1}{\cot \alpha} = \frac{1}{\tan \alpha}$$

$$= \cot\left(\frac{\pi}{2} - \alpha\right);$$

$$\therefore \beta + \gamma = n\pi + \frac{\pi}{2} - \alpha,$$

or

$$\alpha + \beta + \gamma = (2n+1)\frac{\pi}{2}.$$

263. Since $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,

$$\text{the first side} = \frac{\tan^2 A + \tan^2 B + \tan^2 C}{\tan A + \tan B + \tan C}$$

$$\begin{aligned} &= \frac{(\tan A + \tan B + \tan C)^2 - 2 \tan A \tan B - 2 \tan B \tan C - 2 \tan C \tan A}{\tan A + \tan B + \tan C} \\ &= \tan A + \tan B + \tan C - \frac{2(\tan A \tan B + \dots + \dots)}{\tan A \tan B \tan C} \\ &= \tan A + \tan B + \tan C - 2(\cot A + \cot B + \cot C). \end{aligned}$$

264. Let O_1, O_2 be the two points of observation, A and B the two objects, so that $\angle A O_1 O_2 = 45^\circ$, $A O_2 O_1 = O_1 O_2 B = 22\frac{1}{2}^\circ$. Then $\angle O_1 A O_2 = 112\frac{1}{2}^\circ$, $\angle O_1 B O_2 = 22\frac{1}{2}^\circ$, and $O_1 B = O_1 O_2 = 1$ mile.

$$\text{Now from the } \triangle O_1 A O_2, \frac{O_1 A}{1 \text{ mile}} = \frac{\sin 22\frac{1}{2}^\circ}{\sin 112\frac{1}{2}^\circ} = \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1;$$

$$\therefore O_1 A = \sqrt{2} - 1 \text{ miles}; \quad \therefore AB = \sqrt{2} \text{ miles.}$$

Again, if p_1, p_2 be the perpendiculars from AB on $O_1 O_2$

$$p_1 + p_2 = (O_1 A + O_1 B) \sin 45^\circ = AB \sin 45^\circ = 1 \text{ mile.}$$

265. This follows from the identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

where $A + B + C = 180^\circ$, by putting $A = 20^\circ$, $B = 40^\circ$, $C = 120^\circ$.

266. The equation may be written

$$(2 \operatorname{cosec} 2\theta)^3 = 3(2 \operatorname{cosec} 2\theta) + \frac{\cos^3 \theta}{\sin^3 \theta},$$

$$\text{or } \left(\frac{1}{\sin \theta \cos \theta} \right)^3 = \frac{3}{\sin \theta \cos \theta} + \frac{\cos^3 \theta}{\sin^3 \theta};$$

that is,

$$1 = 3 \sin^2 \theta \cos^2 \theta + \cos^6 \theta,$$

which reduces to $(\cos^2 \theta - 1)^3 = 0$, whence $\theta = n\pi$.

267. When

$$A + B + C = 180^\circ,$$

$$1 - \cos^2 B + \cos^2 A - \cos^2 C = \sin^2 B + \sin^2 C - \sin^2 A$$

$$= \sin^2 B + \sin(C+A) \sin(C-A)$$

$$= \sin B \{ \sin(C+A) + \sin(C-A) \},$$

$$\text{since } C+A = 180^\circ - B,$$

$$= 2 \sin B \sin C \cos A.$$

When

$$A + B + C = 0,$$

$$1 - \cos^2 B + \cos^2 A - \cos^2 C = \sin^2 B + \sin(C+A) \sin(C-A)$$

$$= -\sin B \{ \sin(C+A) + \sin(C-A) \},$$

$$\text{since } C+A = -B,$$

$$= -2 \sin B \sin C \cos A.$$

268. We have

$$\cot A - \cot B = \cot B - \cot C,$$

that is

$$\frac{\sin(A-B)}{\sin A \sin B} = \frac{\sin(B-C)}{\sin B \sin C},$$

or

$$\frac{\sin(A-B)}{\sin(B+C)} = \frac{\sin(B-C)}{\sin(A+B)}.$$

$$\text{Whence } \sin(A+B) \sin(A-B) = \sin(B+C) \sin(B-C),$$

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$$

that is,

$$a^2 - b^2 = b^2 - c^2.$$

$$\begin{aligned} 269. \text{ Second side} &= 2 \cos \frac{5\alpha - 2\beta - \gamma}{4} \left\{ \cos \frac{4\beta + 3\gamma - 3\alpha}{4} + \cos \frac{6\beta - 7\gamma + \alpha}{4} \right\} \\ &= \cos \frac{2\alpha + 2\beta + 2\gamma}{4} + \cos \frac{6\beta + 4\gamma - 8\alpha}{4} + \cos \frac{6\alpha + 4\beta - 8\gamma}{4} \\ &\quad + \cos \frac{6\gamma + 4\alpha - 8\beta}{4} \\ &= \cos \frac{\pi}{2} + \cos \left(\frac{3\beta}{2} + \gamma - 2\alpha \right) + \cos \left(\frac{3\alpha}{2} + \beta - 2\gamma \right) \\ &\quad + \cos \left(\frac{3\gamma}{2} + \alpha - 2\beta \right) \\ &= \text{first side.} \end{aligned}$$

270. Denote the radii of the three escribed circles by x, y, z respectively, then we have to shew that

$$(y-z)(z-x)(x-y) + (y-z)(z+x)(x+y) + (z-x)(x+y)(y+z) + (x-y)(y+z)(z+x) = 0.$$

Taking the terms in pairs, the expression on the left reduces to

$$(y-z)\{2(zx+xy)\} + (y+z)\{2(zx-xy)\},$$

$$\text{or } 2x(y-z)(y+z) + 2x(y+z)(z-y),$$

which is identically equal to zero.

271. We have $32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 (\cos 2A - \cos 3A)$.

$$\text{Now } \cos 2A = 2 \cos^2 A - 1 = \frac{2 \times 9}{16} - 1 = \frac{1}{8},$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A = \frac{4 \times 27}{64} - \frac{3 \times 3}{4} = -\frac{9}{16};$$

$$\therefore 32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 \left(\frac{1}{8} + \frac{9}{16} \right) = 11.$$

272. Solving the quadratic, we have $\tan \theta = -1 \pm \sqrt{2}$.

$$\text{Now } \sqrt{2} - 1 = \tan \frac{\pi}{8}. \quad [\text{Art. 251.}]$$

$$-(\sqrt{2} + 1) = -\cot \frac{\pi}{8} = -\tan \left(\frac{\pi}{2} - \frac{\pi}{8} \right).$$

From the first result, we get $\theta = n\pi + \frac{\pi}{8}$,

$$\text{and from the second, } \theta = n\pi - \left(\frac{\pi}{2} - \frac{\pi}{8} \right) = n\pi - \frac{3\pi}{8},$$

both of which are included in $(8n-1)\frac{\pi}{8} \pm \frac{\pi}{4}$.

$$273. \quad (1) \quad 2 \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{7}}{1 - \frac{1}{7^2}} = \tan^{-1} \frac{14}{48} = \tan^{-1} \frac{7}{24} = \cos^{-1} \frac{24}{25},$$

$$\begin{aligned} 4 \tan^{-1} \frac{1}{3} &= 2 \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = 2 \tan^{-1} \frac{3}{4} = \tan^{-1} \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \tan^{-1} \frac{24}{7} \\ &= \sin^{-1} \frac{24}{25}. \end{aligned}$$

Thus each side of the identity $= \frac{24}{25}$.

$$(2) \quad \frac{3 \sin 2\alpha}{5 + 3 \cos 2\alpha} = \frac{6 \tan \alpha}{1 + \tan^2 \alpha} \div \left\{ 5 + \frac{3(1 - \tan^2 \alpha)}{1 + \tan^2 \alpha} \right\} = \frac{3 \tan \alpha}{4 + \tan^2 \alpha};$$

$$\therefore \text{first side of the identity} = \tan^{-1} \frac{3 \tan \alpha}{4 + \tan^2 \alpha} + \tan^{-1} \left(\frac{\tan \alpha}{4} \right)$$

$$\begin{aligned} &= \tan^{-1} \frac{\frac{3 \tan \alpha}{4 + \tan^2 \alpha} + \frac{\tan \alpha}{4}}{1 - \frac{3 \tan^2 \alpha}{4(4 + \tan^2 \alpha)}} = \tan^{-1} \frac{16 \tan \alpha + \tan^3 \alpha}{16 + \tan^2 \alpha} \\ &= \tan^{-1} (\tan \alpha) = \alpha. \end{aligned}$$

274. We have

$$\frac{s(s-a)}{bc} = \frac{b^2+c^2}{4bc};$$

$$\therefore 2s(2s-2a) = b^2+c^2;$$

or

that is,

$$(b+c+a)(b+c-a) = b^2+c^2,$$

$$b^2+c^2+2bc-a^2 = b^2+c^2;$$

$$\therefore \frac{a^2}{2} = bc,$$

which proves the proposition.

275. We have $\sin B = \frac{b}{a} \sin A = \frac{119 \sin 50^\circ}{97}$,

$$\log 119 = 2.0755470$$

$$\log \sin 50^\circ = 1.8842540$$

$$1.9598010$$

$$\log 97 = 1.9867717$$

$$\log \sin B = 1.9730293$$

diff. for $1' = 460$,

$$\log \sin 70^\circ = \frac{1.9729858}{435}$$

$$\frac{435}{460} \times 60'' = 57'';$$

$\therefore B = 70^\circ 0' 57''$, or $109^\circ 59' 3''$, both values being admissible since $a < b$;

$\therefore C = 59^\circ 59' 3''$ or $20^\circ 0' 57''$.

276. The $\angle BD_1F_1 = \frac{1}{2}$ (suppt of $\angle F_1BD_1$) $= \frac{B}{2} = \angle BF_1D_1$.

Similarly

$$\angle CD_1E_1 = \frac{C}{2};$$

$$\therefore \angle F_1D_1E_1 = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}.$$

Again from the isosceles triangle $A E_1 F_1$,

$$\angle AF_1E_1 = 90^\circ - \frac{A}{2}; \quad \therefore \angle D_1F_1E_1 = 90^\circ - \frac{A+B}{2} = \frac{C}{2},$$

and similarly

$$\angle D_1E_1F_1 = \frac{B}{2}.$$

$$\text{Now } r_a = \frac{E_1F_1 \sin \frac{1}{2} E_1 \sin \frac{1}{2} F_1}{\cos \frac{1}{2} D_1} = \frac{2s \sin \frac{A}{2} \sin \frac{B}{4} \sin \frac{C}{4}}{\cos \left(45^\circ + \frac{A}{4}\right)}$$

$$= 4s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{\cos \frac{A}{4}}{\sqrt{2} \left(\cos \frac{A}{4} - \sin \frac{A}{4}\right)}$$

$$= 4\sqrt{2s} \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{1}{1 - \tan \frac{A}{4}};$$

$$\begin{aligned}\therefore \frac{1}{r_a} : 1 - \tan \frac{A}{4} &= 1 : 4\sqrt{2s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4}} \\ &= \frac{1}{r_b} : 1 - \tan \frac{B}{4} = \frac{1}{r_c} : 1 - \tan \frac{C}{4}\end{aligned}$$

by symmetry.

277. The expression $= \tan^{-1} \frac{x \cos \theta}{1 - x \sin \theta} - \tan^{-1} \frac{x - \sin \theta}{\cos \theta}$

and this reduces to $\tan^{-1} \left\{ \frac{\sin \theta (1 - 2x \sin \theta + x^2)}{\cos \theta (1 - 2x \sin \theta + x^2)} \right\}$,

which equals $\tan^{-1} (\tan \theta)$, or θ .

278. By Example XIII. c. 8 we have

$$\frac{\cos A + \cos B}{4 \sin^2 \frac{C}{2}} = \frac{a+b}{2c};$$

$\therefore a+b=2c$; whence a, c, b are in A.P.

279. The expression

$$\begin{aligned}&= 2 \cos \alpha \cos \beta \{ \cos(\gamma+\delta) + \cos(\gamma-\delta) \} + 2 \sin \alpha \sin \beta \{ \cos(\gamma-\delta) - \cos(\gamma+\delta) \} \\ &= 2 \cos(\gamma+\delta) \cos(\alpha+\beta) + 2 \cos(\gamma-\delta) \cos(\alpha-\beta) \\ &= \cos(\alpha+\beta+\gamma+\delta) + \cos(\alpha+\beta-\gamma-\delta) + \cos(\alpha-\beta+\gamma-\delta) + \cos(\alpha-\beta-\gamma+\delta).\end{aligned}$$

280. The $\angle BIC = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}$;

$$\therefore \rho_1 = \frac{a}{2 \sin BIC} = \frac{a}{2 \cos \frac{A}{2}};$$

$$\therefore \rho_1 \rho_2 \rho_3 = \frac{abc}{8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{R^3 \sin A \sin B \sin C}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 2rR^2,$$

since $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

281. This is a particular case of Ex. 13 in XVII. a.

282. The equation may be written

$$b^2 \sin^2 2\theta = (c - a \cos 2\theta)^2,$$

or $b^2(1 - \cos^2 2\theta) = c^2 - 2ac \cos 2\theta + a^2 \cos^2 2\theta,$

that is, $(a^2 + b^2) \cos^2 2\theta - 2ac \cos 2\theta + c^2 - b^2 = 0;$

therefore, by the theory of quadratic equations,

$$\cos 2\alpha + \cos 2\beta = \frac{2ac}{a^2 + b^2};$$

$$\therefore 2 \cos^2 \alpha - 1 + 2 \cos^2 \beta - 1 = \frac{2ac}{a^2 + b^2};$$

whence $\cos^2 \alpha + \cos^2 \beta = \frac{a^2 + ac + b^2}{a^2 + b^2}.$

283. We have $c^2 = a^2 + b^2 - 2ab \cos C$

$$= 2 + 2 + \sqrt{2} - 2\sqrt{2}\sqrt{2 + \sqrt{2}} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$= 4 + \sqrt{2} - \sqrt{2}(2 + \sqrt{2}) = 2 - \sqrt{2};$$

$$\therefore c = \sqrt{2 - \sqrt{2}}.$$

Now $\sin A = \frac{a}{c} \sin C = \frac{\sqrt{2}}{\sqrt{2 - \sqrt{2}}}, \quad \frac{\sqrt{2 - \sqrt{2}}}{2} = \frac{1}{\sqrt{2}};$

therefore $A = 45^\circ$, or 135° , and since a is not the greatest side the smaller value must be taken.

Therefore

$$B = 112\frac{1}{2}^\circ.$$

284. We have $\sin 3A = 3 \sin A - 4 \sin^3 A;$

$$\therefore \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin^3 A;$$

$$\therefore \Sigma \sin^3 A = \frac{3}{4} \Sigma \sin A - \frac{1}{4} \Sigma \sin 3A,$$

Now $\Sigma \sin A = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$

and $\Sigma \sin 3A = 2 \sin \frac{3(A+B)}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$

$$= 2 \sin \left(270^\circ - \frac{3C}{2} \right) \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$= -2 \cos \frac{3C}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$\begin{aligned}
 &= 2 \cos \frac{3C}{3} \left\{ \sin \frac{3C}{2} - \cos \frac{3(A-B)}{2} \right\} \\
 &= 2 \cos \frac{3C}{2} \left\{ -\cos \frac{3(A+B)}{2} - \cos \frac{3(A-B)}{2} \right\} \\
 &= -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}; \\
 \therefore \Sigma \sin^3 A &= 3 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} - \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}.
 \end{aligned}$$

285. Take the third figure on p. 131, and first suppose that

$$\angle A B_2 C = 2 \angle A B_1 C.$$

Then it easily follows that $\triangle C B_1 B_2$ is equilateral;

$$\therefore b \sin A = a \sin B \text{ becomes } b \sin A = \frac{\sqrt{3}}{2} a.$$

Secondly, suppose that $\angle A C B_1 = 2 \angle A C B_2$.

$$\begin{aligned}
 \text{Then } \angle A C B_2 &= \angle B_1 - \angle A = \angle B_2 C B_1 = 180^\circ - 2 \angle B_1; \\
 &\therefore 3B_1 = 180^\circ + A,
 \end{aligned}$$

$$\sin 3B_1 + \sin A = 0; \text{ or } 3 \sin B_1 - 4 \sin^3 B_1 + \sin A = 0.$$

Substituting $\frac{b}{a} \sin A$ for $\sin B_1$, and reducing we obtain the required result.

286. If we write \sqrt{y} in the place of x the resulting equation has $\tan^2 \alpha$, $\tan^2 \beta$, $\tan^2 \gamma$ for its roots. If in the last equation we further write z for $1+y$ the resulting equation has $\sec^2 \alpha$, $\sec^2 \beta$, $\sec^2 \gamma$ for its roots.

After making the above substitutions the equation in z is

$$\begin{aligned}
 z^3 - (p^2 + 3) z^2 + (4p^2 - 2pr + 3) z - (p - r)^2 - 1 &= 0; \\
 \therefore \sec^2 \alpha \sec^2 \beta \sec^2 \gamma &= \text{product of the roots} \\
 &= (p - r)^2 + 1.
 \end{aligned}$$

Otherwise. Let t_1 , t_2 , t_3 be the roots of the given equation; then

$$\begin{aligned}
 \sec^2 \alpha \sec^2 \beta \sec^2 \gamma &= (1 + t_1^2)(1 + t_2^2)(1 + t_3^2) \\
 &= 1 + \sum t_1^2 + \sum t_1^2 t_2^2 + t_1^2 t_2^2 t_3^2,
 \end{aligned}$$

where

$$\Sigma t_1 = p, \quad \Sigma t_1 t_2 = 0, \quad t_1 t_2 t_3 = r.$$

Thus

$$\sec^2 \alpha \sec^2 \beta \sec^2 \gamma = 1 + p^2 - 2pr + r^2.$$

287. Square the given equation, and write it in the form

$$\begin{aligned}
 \sin^2 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \sin^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \sin^2 \left(\frac{\pi}{4} - \frac{\gamma}{2} \right) \\
 = \cos^2 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \cos^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cos^2 \left(\frac{\pi}{4} - \frac{\gamma}{2} \right),
 \end{aligned}$$

or $(1 - \sin \alpha)(1 - \sin \beta)(1 - \sin \gamma) = (1 + \sin \alpha)(1 + \sin \beta)(1 + \sin \gamma);$
 $\therefore \sin \alpha + \sin \beta + \sin \gamma + \sin \alpha \sin \beta \sin \gamma = 0,$

or $4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} + 8 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} = 0;$

hence $1 + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 0,$

that is, $1 + \frac{1}{2}(\cos \alpha + \cos \beta + \cos \gamma - 1) = 0;$
 $\therefore \cos \alpha + \cos \beta + \cos \gamma + 1 = 0.$

288. Each side of the heptagon subtends an angle $\frac{\pi}{7}$ at the circumference of the circle whose diameter is 2. Therefore if x represent a side,

$$x = 2 \sin \frac{\pi}{7}.$$

Now by Art. 331, the roots of the equation

$$8y^3 + 4y^2 - 4y - 1 = 0,$$

are $\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}.$

Therefore $2 \cos \frac{2\pi}{7}$ satisfies $y^3 + y^2 - 2y - 1 = 0.$

Put $y = 2 - 4 \sin^2 \frac{\pi}{7} = 2 - x^2$ in this equation.

We obtain, after reduction,

$$x^6 - 7x^4 + 14x^2 - 7 = 0,$$

the roots of which are $2 \sin \frac{\pi}{7}, 2 \sin \frac{2\pi}{7}, 2 \sin \frac{3\pi}{7}.$

The first of these values corresponds to a side of the heptagon, the second and third to chords subtending at the circumference angles of $\frac{2\pi}{7}$ and $\frac{3\pi}{7}$ respectively. That is they represent the diagonals of the heptagon, as is easily seen from a figure.

289. We have $\cot(A + C) = -\cot B = -1;$

$$\therefore \frac{\cot A \cot C - 1}{\cot A + \cot C} = -1;$$

that is, $1 + \cot A + \cot C + \cot A \cot C = 2,$

or $(1 + \cot A)(1 + \cot B) = 2.$

293. We have $\sin \frac{\pi}{14} = \cos \left(\frac{\pi}{2} - \frac{\pi}{14} \right) = \cos \frac{3\pi}{7}$,

which is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

See solution of XXV. c. Ex. 16.

294. The distances of the successive heaps from the starting point are

$$2r \sin \frac{\pi}{n}, \quad 2r \sin \frac{2\pi}{n}, \quad 2r \sin \frac{3\pi}{n}, \dots, 2r \sin \frac{(n-1)\pi}{n};$$

∴ the whole distance traversed is twice the sum of this series.

$$\text{Now the sum of the sines} = \frac{\sin \frac{(n-1)\pi}{2n} \sin \frac{1}{2} \left\{ \frac{\pi}{n} + \frac{(n-1)\pi}{n} \right\}}{\sin \frac{\pi}{2n}} \quad [\text{Art. 296.}]$$

$$= \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2n} \right) \sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} = \cot \frac{\pi}{2n},$$

whence the required result follows.

295. $\cos \frac{\pi}{15} \cos \frac{4\pi}{15} = \frac{1}{2} \left(\cos \frac{\pi}{3} + \cos \frac{\pi}{5} \right) = \frac{1}{2} \left(\frac{1}{2} + \frac{\sqrt{5}+1}{4} \right) = \frac{3+\sqrt{5}}{8};$

$$\cos \frac{2\pi}{15} \cos \frac{7\pi}{15} = \frac{1}{2} \left(\cos \frac{\pi}{3} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left(\frac{1}{2} - \frac{\sqrt{5}-1}{4} \right) = \frac{3-\sqrt{5}}{8};$$

$$\cos \frac{3\pi}{15} \cos \frac{6\pi}{15} = \frac{1}{2} \left(\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{1}{4};$$

$$\cos \frac{5\pi}{15} = \cos \frac{\pi}{3} = \frac{1}{2};$$

∴ multiplying these results together, we have

$$\frac{1}{8} \left(\frac{3+\sqrt{5}}{8} \right) \left(\frac{3-\sqrt{5}}{8} \right) = \frac{4}{8^3} = \frac{2^2}{2^9} = \left(\frac{1}{2} \right)^7.$$

296. We have $ax + by + cz = 2\Delta$.

Now $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2$
 $= (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2;$

or $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 4\Delta^2 = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2.$

∴ $x^2 + y^2 + z^2$ is a minimum when the expression on the right is zero; that is when

$$bx = ay, \quad cy = bz, \quad az = cx;$$

$$\therefore \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}.$$

297. With the notation and fig. of Art. 231, suppose AD and BC intersect in E , then r_a is the radius of the escribed circle opposite to E in the $\triangle ABE$;

$$\therefore r_a = \frac{a \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{A+B}{2}},$$

$$\therefore \frac{a}{r_a} = \tan \frac{A}{2} + \tan \frac{B}{2};$$

$$\therefore \frac{a}{r_a} + \frac{c}{r_c} = \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} + \tan \frac{D}{2}$$

$$= \frac{b}{r_b} + \frac{d}{r_d}, \text{ similarly.}$$

298. Draw AH, AH' perpendicular to BC ; then

$$\angle PAH = (90^\circ - B) - (90^\circ - C) = C - B,$$

and

$$AH = 2R \sin B \sin C,$$

$$\therefore AP = \frac{2R \sin B \sin C}{\cos(C-B)};$$

$$\therefore \frac{1}{AP} + \dots + \dots = \frac{1}{2R} \left\{ \frac{\cos(B-C)}{\sin B \sin C} + \dots + \dots \right\}$$

$$= \frac{1}{4R} \cdot \frac{(\sin 2C + \sin 2B) + \dots + \dots}{\sin A \sin B \sin C}$$

$$= \frac{2}{R}. \quad [\text{Art. 135, Ex. 1.}]$$

Again, $BA' = 2R \cos BA'A = 2R \cos C$, since B, A', A, C are concyclic,

$$\therefore A'H' = 2R \cos B \cos C;$$

$$\therefore \frac{1}{A'P} + \dots + \dots = \frac{1}{2R} \left\{ \frac{\cos(C-A)}{\cos B \cos C} + \dots + \dots \right\}$$

$$= -\frac{1}{2R} \frac{(\cos 2C + \cos 2B) + \dots + \dots}{\cos A \cos B \cos C}$$

$$= -\frac{2(\cos 2A + \cos 2B + \cos 2C)}{4R \cos A \cos B \cos C}$$

$$= \frac{1}{2R} \cdot \frac{4 \cos A \cos B \cos C + 1}{\cos A \cos B \cos C}. \quad [\text{XII. d. Ex. 9.}]$$

299. Let $\tan \frac{\theta}{2} = t$, then $\sin \theta = \frac{2t}{1+t^2}$, $\cos \theta = \frac{1-t^2}{1+t^2}$; substitute these values in the given equation; then after reduction, we obtain

$$t^4 b + 2t^3(c-a) - 2t(c+a) - b = 0.$$

This equation has four roots; three of which are

$$\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}, \tan \frac{\gamma}{2};$$

also

$$t_1 t_2 t_3 t_4 = -1, \quad t_2 t_3 + t_3 t_4 + \dots = 0.$$

Eliminating t_4 by means of these equations,

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = \frac{1}{t_2 t_3} + \frac{1}{t_3 t_1} + \frac{1}{t_1 t_2};$$

$$\begin{aligned} \therefore \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} + \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \\ = \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2}; \end{aligned}$$

that is,

$$\Sigma \left(\tan \frac{\beta}{2} \tan \frac{\gamma}{2} - \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \right) = 0;$$

$$\therefore \Sigma \frac{\sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}}{\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}} = 0;$$

$$\therefore \Sigma \cos \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 0;$$

$$\therefore \Sigma (\cos \beta + \cos \gamma) \sin \alpha = 0;$$

or

$$\Sigma \sin(\alpha + \beta) = 0.$$

300. By Example 2, Art. 331,

$$\sec^2 \frac{\pi}{7}, \quad \sec^2 \frac{2\pi}{7}, \quad \sec^2 \frac{3\pi}{7}$$

are roots of the equation

$$x^3 - 24x^2 + 80x - 64 = 0;$$

$$\therefore \sec^2 \frac{\pi}{7} + \sec^2 \frac{2\pi}{7} + \sec^2 \frac{3\pi}{7} = 24.$$

Also from XXV. c. Ex. 21,

$$\operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8,$$

whence the required result follows.

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